

Black Holes, Ellipsoids, and Nonlinear Waves in Pseudo-Finsler Spaces and Einstein Gravity

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Abstract

We model pseudo-Finsler geometries, with pseudo-Euclidean signatures of metrics, for two classes of four dimensional nonholonomic manifolds: a) tangent bundles with two dimensional base manifolds and b) pseudo-Riemannian/ Einstein manifolds. Such spacetimes are enabled with nonholonomic distributions and associated nonlinear connection structures and theirs metrics are solutions of the field equations in general relativity or in generalized gravity theories with nonholonomic variables. We rewrite the Schwarzschild metric in Finsler variables and use it for generating new classes of locally anisotropic black holes and (or) stationary deformations to ellipsoidal configurations. There are analyzed the conditions when such metrics describe imbedding of black hole solutions into nontrivial solitonic backgrounds.

Keywords: Pseudo-Finsler geometry, nonholonomic manifolds and bundles, nonlinear connections, black holes and ellipsoids.

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1 Introduction

The goal of this work is to construct new classes of (Finsler) black hole solutions and analyze their deformations to metrics with ellipsoidal symmetry and (or) imbedding into nontrivial solitonic backgrounds. Such models of pseudo-Finsler spacetimes can be elaborated on tangent bundles/pseudo-Riemannian manifolds enabled with corresponding nonholonomic frame structures. Let us motivate our interest in this problem:

Black holes are investigated in great depth and detail for more than fifty years for all important gravity theories like general relativity and string/brane gravity and their bimetric, gauge, noncommutative modifications/generalizations etc; we cite here monographs [1, 2, 3, 4, 5] and a recent resource letter [6] for literature on black hole physics and mathematics. Various classes of locally anisotropic exact solutions (describing generalized Finsler metrics) were constructed in low and extra dimensional gravity for spacetime models with nonholonomic distributions, see reviews of results and methods in Refs. [7, 8, 9, 10] (see also examples of 3, 4 and 5 dimensional locally anisotropic black hole/ ellipsoid solutions Refs. [11, 12, 13, 17, 14, 15, 16, 18, 19]). Nevertheless, the mentioned types of locally anisotropic solutions are not exactly for the pseudo-Finsler spacetimes but for more general nonholonomic configurations. Up till present, there were not published works on exact solutions for black hole metrics and connections in Finsler gravity models.

A surprising recent result is that for certain classes of nonholonomic distributions/ frames we can model (pseudo) Lagrange and Finsler like geometries on (pseudo) Riemannian manifolds and when the metric and connection structures can be constrained to solve usual Einstein gravitational equations in general relativity [7, 8, 18, 19], or their generalizations [9, 10, 13, 17]. So, the task to construct Finsler black hole solutions is not only a formal one related to non-Riemannian spacetimes but also presents a substantial interest in modelling locally anisotropic black hole configurations, with generic off-diagonal metrics, in Einstein gravity. Here we emphasize that locally anisotropic/ nonholonomic spacetimes (Finsler like and more general ones) defined as exact solutions of standard Einstein equations are not subjected to experimental constraints as in the case of modified gravity theories constructed on (co) tangent bundles [20, 21].

Fixing a nonholonomic distribution defined by a Finsler fundamental function on a (pseudo) Riemannian manifold, we can introduce Finsler type (and their almost Kähler analogs) variables and consider associated nonlinear connection structures. For any metric tensor, we can construct an

infinite number of metric compatible linear connections completely defined by the coefficients of the same metric but stating different nonholonomic spacetime configurations (for instance, with 2+2 dimensional splitting of dimension). Such Finsler–Kähler and other types of nonholonomic variables are convenient for elaborating different perturbative and nonperturbative methods and deformation/ brane quantizations of Einstein gravity and models of gauge gravity [22, 23, 24, 25, 26, 27, 28, 29]. Nonholonomic Finsler variables can be also considered for constructing exact solutions in Einstein gravity; any geometric/ physical object and fundamental equations can be written in such variables. This provides us various possibilities to redefine important physical solutions in gravity theories with local anisotropy and nonholonomic distributions, in particular for (pseudo) Finsler spaces and nonholonomic Einstein manifolds.

It is complicated to find new classes of black hole solutions both in Einstein gravity and modified gravity theories. Any type of such solutions presents a substantial interest for possible applications in modern astrophysics and cosmology. The subject of pseudo–Finsler black holes is also interesting for various studies in mathematical physics as it is connected to new methods of constructing exact solutions defining spacetimes with generalized symmetries and nonlinear gravitational interactions [7, 8, 10].

Recently, a series of works on Finsler analogous of gravity and applications [30, 31, 32, 33, 34] were published following various purposes in modern cosmology [35, 36, 37, 38], string gravity [39, 40, 41] and quantum gravity [42, 22, 23, 24], and alternative gravity theories and physical applications of nonholonomic Ricci flows [25, 43]; see also monographs [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55], and references therein, on “early” physical models with Finsler geometries. It is an important task to investigate if certain Finsler like gravitational models (commutative and noncommutative ones, nonsymmetric metric generalizations etc) may have, or not, black hole type solutions and to understand when some (pseudo) Finsler metrics can be related to modern/ standard theories of gravity.

Exact solutions in gravity theories, including black hole metrics, carry a great deal of information the gravitational theories themselves. They can be considered both for theoretical and, in many cases, experimental tests of physical models. In our approach, we construct new classes of black hole/ ellipsoid solutions generalizing similar ones in Einstein gravity, or being Schwarzschild analogs in pseudo–Finsler spacetimes, and analyze possible physical implications.

In this article, we present and discuss the main properties of Finsler black hole solutions generated following the so-called anholonomic frame

method, elaborated and developed in our previous works [7, 8, 9, 10, 11, 12, 13, 17, 14, 15, 16, 18, 19, 35]. We shall work explicitly with Finsler spacetime configurations, and their nonholonomic transforms, but not with generalized locally anisotropic and/or Lagrange–Finsler structures in the bulk of our former constructions. Readers are recommended to study preliminary the reviews [7, 8] on applications in physics of the geometry of nonlinear connections and associated nonholonomic frames and modelling of Lagrange–Finsler geometries on (pseudo) Riemannian spacetimes.

We emphasize that in this work the Finsler geometry models are elaborated for the canonical distinguished connection following geometric methods developed in Refs. [50, 52, 8, 10]. In our approach, we usually do not work with the Chern connection [55] (sometimes, called also Rund’s connection [45]) for Finsler spaces because this connection is metric noncompatible, which results in a more sophisticated mathematical formalism and have less physical motivations from viewpoint of standard gravity theories, see discussions in Ref. [8] and Introduction to [10]. One should be noted here that applying the anholonomic frame method it is also possible to construct locally anisotropic black hole solutions in metric–affine [56] generalizations of (non)commutative Lagrange–Finsler gravities, as it is provided in Parts I and III of Ref. [10]. Nevertheless, for simplicity, this paper is oriented to keep the geometric and physical constructions more closed to the Einstein gravity, in Finsler like variables, and four dimensional pseudo–Finsler analogs on nonholonomic tangent bundles.

The paper is organized as follows:

In section 2, we outline some basic formulas and conventions on modeling Finsler geometries on tangent bundles and nonholonomic (pseudo) Riemannian manifolds. We state the metric ansatz and construct in general form, applying the anholonomic frame method, a class of exact solutions defining nonholonomic Einstein and Finsler spaces, see section 3.

Pseudo–Finsler generalizations of the Schwarzschild solution and nonholonomic ellipsoidal deformations of Einstein metrics are presented in section 4. We discuss there how such metrics can be constructed on tangent bundles/ Einstein manifolds. There are provided examples when black hole solutions can be imbedded and/or nonholonomically mapped on pseudo–Finsler spaces and/or deformed nonholonomically into exact solutions of Einstein equations in general relativity.

In section 5, we discuss the obtained solutions for pseudo–Finsler and Einstein spaces and formulate conclusions. For convenience, in Appendix, we outline the main concepts and formulas on pseudo–Finsler geometry.

Finally, we note that we provide references only to a series of recent

Finsler works which may have implications for standard theories of gravity and high energy physics, in the spirit of reviews [8, 7] and monograph [10]. On applications of nonholonomic geometry and Lagrange–Finsler geometry methods in modern quantum gravity, one should be consulted Refs. [22, 23, 24, 25, 26, 27, 28, 29]. More details on the geometry of Finsler spaces, generalizations and "nonstandard" applications in physics are presented in [44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55] and references therein.

2 Pseudo–Finsler Bundles and Einstein–Finsler Manifolds

Let us consider a four dimensional (4-d) manifold \mathbf{V} of necessary smooth class (in brief, we shall use the terms space, or spacetime, for corresponding positive/ negative signatures of metrics). Such spacetimes can be enabled with a conventional $2+2$ splitting (defined by a nonholonomic, equivalently, anholonomic/nonintegrable, distribution), when local coordinates $u = (x, y)$ on an open region $U \subset \mathbf{V}$ are labelled in the form $u^\alpha = (x^i, y^a)$, with indices of type $i, j, k, \dots = 1, 2$ and $a, b, c, \dots = 3, 4$. For tensor like objects on \mathbf{V} , their coefficients will be considered with respect to a general (non–coordinate) local basis $e_\alpha = (e_i, e_a)$.

If $\mathbf{V} = TM$ is the total space of a tangent bundle (TM, π, M) on a two dimensional (2-d) base manifold M , the values x^i and y^a are respectively the base coordinates (on a low–dimensional space/ spacetime) and fiber coordinates (velocity like). In such a case, the geometric constructions and physical models will be performed on tangent bundles which results in various types of generalizations/ violations of the local Lorentz invariance, non–Riemannian locally anisotropic gravitational effects etc.

Alternatively, we can consider that $\mathbf{V} = V$ is a 4–d nonholonomic manifold (in particular, a pseudo–Riemannian one) with local fibered structure.¹ Following this approach, we can model various types of gravitational interactions/ configurations characterizing nonlinear and/or nonholonomic effects in general relativity.² In such a case, we shall treat x^i and y^a , respectively, as conventional horizontal/ nonholonomic (h) and vertical / holonomic (v) coordinates (both types of such coordinates can be time– or space–like ones).

Primed (double primed, underlined etc) indices, for instance $\alpha' = (i', a')$,

¹A pair $(\mathbf{V}, \mathcal{N})$, where \mathbf{V} is a manifold and \mathcal{N} is a nonintegrable distribution on \mathbf{V} , is called a nonholonomic manifold.

²In this work, boldface symbols will be used for nonholonomic manifolds/ bundles (nonholonomic spaces) and geometric objects on such spaces.

$\beta'' = (j'', b'')$, $\underline{\gamma} = (\underline{k}, \underline{c})$, ... will be used for labelling coordinates with respect to a different local basis $e_{\alpha'} = (e_{i'}, e_{a'})$, or its dual $e^{\alpha'} = (e^{i'}, e^{a'})$. For convenience, we provide a summary on (pseudo) Finsler geometry and Finsler variables on (pseudo) Riemannian spaces in Appendix, see details in Refs. [8, 7, 10].

2.1 (Pseudo) Finsler/ Riemannian metrics and Finsler variables

Let us consider a nonholonomic tangent bundle/ manifold, \mathbf{V} endowed with a metric structure $\mathbf{g} = \{\mathbf{g}_{\alpha\beta}\}$ of arbitrary signature $(\pm 1, \pm 1, \pm 1, \pm 1)$.³ For physical applications, we shall chose any convenient orientation of three space and one time like local coordinates.

The coefficients of a general (pseudo) Riemannian metric $g = g_{\alpha'\beta'} e^{\alpha'} \otimes e^{\beta'}$, for $e^{\alpha'} = (e^{i'}, e^{a'}) = e^{\alpha'}_{\alpha}(u) du^{\alpha}$, can be parametrized in a form adapted to a nonholonomic 2 + 2 splitting induced by a Finsler generating function, see explanations of formulas (A.1) in Appendix. We write

$$\begin{aligned} \mathbf{g} &= g_{i'j'}(u) e^{i'} \otimes e^{j'} + h_{a'b'}(u) \mathbf{e}^{a'} \otimes \mathbf{e}^{b'}, \\ e^{a'} &= \mathbf{e}^{a'} + N_{i'}^{a'}(u) e^{i'}, \end{aligned} \quad (1)$$

when the values $\mathbf{g}_{\alpha'\beta'} = [g_{i'j'}, h_{a'b'}]$ are related by transforms

$$\mathbf{g}_{\alpha'\beta'} e^{\alpha'}_{\alpha} e^{\beta'}_{\beta} = \mathbf{f}_{\alpha\beta} \quad (2)$$

³We emphasize that the spacetime signature may be encoded formally into certain systems of frame (vielbein) coefficients and coordinates, some of them being proportional to the imaginary unity i , when $i^2 = -1$. For instance, on a local tangent Minkowski space of signature $(-, +, +, +)$, we can chose $e_{0'} = i\partial/\partial u^{0'}$, where i is the imaginary unity, $i^2 = -1$, and write $e_{\alpha'} = (i\partial/\partial u^{0'}, \partial/\partial u^{1'}, \partial/\partial u^{2'}, \partial/\partial u^{3'})$. To consider such formal local Euclidean coordinates was largely used in the past in a number of textbooks on relativity theory (see, for instance, [57, 58]) which is useful for some purposes of analogous modelling of gravity theories as effective Lagrange mechanics, or Finsler like, geometries, but this does not mean that we work with a complexification of classical spacetimes, see discussion in Ref. [8]. If such formal complex coordinates and frame components are introduced into consideration, we can use respectively the terms pseudo-Euclidean, pseudo-Riemannian, pseudo-Finsler spaces etc. The term "pseudo-Finsler" (considered in this paper) was also used more recently for some analogous gravity like models (see, for instance, [34]). Nevertheless, it should be noted here that following formal Finsler constructions on real manifolds when some frames/coordinates contain the imaginary unity, we can elaborate geometric and physical models with various types of signature. Such geometric structures were presented in direct, or indirect, form in the bulk of monographs on Lagrange-Finsler geometry and applications [45, 46, 47, 48, 49, 50, 51, 53, 54, 55, 10] even geometers preferred to work, for simplicity, with the positive signature.

to a (pseudo-Finsler) metric $\mathbf{f}_{\alpha\beta} = [f_{ij}, f_{ab}]$ (A.10) and corresponding N-adapted dual canonical basis ${}^c e^\alpha = (dx^i, {}^c e^a)$. Considering any given values $\mathbf{g}_{\alpha'\beta'}$ and $\mathbf{f}_{\alpha\beta}$, we have to solve a system of quadratic algebraic equations (2) with unknown variables $e^{\alpha'}$. How to define in explicit form such frame coefficients (vierbeins) and coordinates we discuss in Refs. [7, 26]. For instance, in general relativity, there are 6 independent values $\mathbf{g}_{\alpha'\beta'}$ and up till ten nonzero induced coefficients $\mathbf{f}_{\alpha\beta}$ allowing us to define always a set of vierbein coefficients $e^{\alpha'}$. Usually, for certain formally diagonalized representations of h- and v-components of metrics, a subset of such coefficients can be taken to be zero, for given values $[g_{i'j'}, h_{a'b'}, N_{i'}^{a'}]$ and $[f_{ij}, f_{ab}, {}^c N_i^a]$, when

$$N_{i'}^{a'} = e_{i'}^i e_a^{a'} {}^c N_i^a \quad (3)$$

for $e_{i'}^i$ being inverse to $e^{i'}$.

We emphasize that using nonholonomic 2+2 distributions stated by values $e^{\alpha'} = (e_{i'}^i, e_a^{a'}, \dots)$, with associated N-connection coefficients $N_{i'}^{a'}$, subjected to conditions (2) and (3), we can transform any (pseudo) Riemannian configuration with metric $\mathbf{g}_{\alpha'\beta'}$ into a (pseudo) Finsler configuration $(f_{ij}, f_{ab}, {}^c N_i^a)$ determined by a fundamental Finsler function $F(u)$. For any given data $(\mathbf{g}_{\alpha'\beta'}, N_{i'}^{a'})$, there is a 2+2 vierbein system when the geometric objects/ variables on a (pseudo) Riemannian spacetime are encoded into geometric objects/ variables of a (pseudo) Finsler geometry modelled effectively by a generating function $F(u)$. Inversely, it is possible to model any (pseudo) Finsler space ${}^2F(M, F(x, y))$ as a nonholonomic (pseudo) Riemannian manifold with some $\mathbf{g}_{\alpha'\beta'} = [g_{i'j'}, h_{a'b'}]$ if we fix a tetradic structure defining a nonholonomic 2+2 distribution transforming $\mathbf{f}_{\alpha\beta}$ into $\mathbf{g}_{\alpha'\beta'}$.

A (pseudo) Riemannian geometry is completely defined by one fundamental geometric object which is the metric structure \mathbf{g} . It determines a unique metric compatible and torsionless Levi-Civita connection ∇ . In order to construct a model of (pseudo) Finsler geometry, we need a generating fundamental Finsler function $F(x, y)$ which in certain canonical approaches generates three fundamental geometric objects: 1) a N-connection \mathbf{N} , 2) a Sasaki type metric $\mathbf{f}_{\alpha\beta}$ and 3) a d-connection \mathbf{D} , see definitions in Appendix. For purposes of this article, it is convenient to work with the canonical N-connection ${}^c \mathbf{N} = \{ {}^c N_i^a \}$ (A.8), d-metric $\mathbf{f}_{\alpha\beta}$ (A.10) and the canonical d-connection $\hat{\mathbf{D}}$ (A.25), all uniquely defined by F , and to perform some necessary vierbein transform of such geometric objects. In brief, we can say that a (pseudo) Riemannian geometry is characterized by data (\mathbf{g}, ∇) and a (pseudo) Finsler geometry is characterized by data $(F, \mathbf{f}, \mathbf{N}, \mathbf{D})$, with any general \mathbf{N} and \mathbf{D} . Such models can be constructed on tangent bundles or

on N-anholonomic (pseudo) Riemannian manifolds. In a (pseudo) Finsler case, it is involved a more "rich" geometric structure with three fundamental geometric objects: metric, N-connection and linear connection (in general, both types of connections can be general ones, not obligatory generated by a fundamental Finsler function).

Using distortions of d-connections, $\nabla = \hat{\mathbf{D}} + \text{ }_i\mathbf{Z}$ (A.26), and nonholonomic frame transform (2) and (3), we can encode a canonical (pseudo) Finsler geometry $(F, \mathbf{f}, \text{ }^c\mathbf{N}, \hat{\mathbf{D}})$ into a (pseudo) Riemannian configuration (\mathbf{g}, ∇) , with \mathbf{g} induced by a generic off-diagonal metric ansatz for \mathbf{f} , see (A.12). Inversely, for any (pseudo) Riemannian space, we can chose any nonholonomic 2+2 distribution induced by a necessary Finsler type generating function $F(x, y)$ and express any data (\mathbf{g}, ∇) into, for instance, some canonical (pseudo) Finsler ones. We say that we introduce (nonholonomic) Finsler variables on a (pseudo) Riemannian manifold and deform nonholonomically the linear connection structure, $\nabla \rightarrow \hat{\mathbf{D}}$ in order to model a Finsler geometry by a corresponding nonholonomic distribution. For such geometric constructions, we can work equivalently with two types of linear connections, ∇ and/or $\hat{\mathbf{D}}$, because all values $\nabla, \hat{\mathbf{D}}$ and $\text{ }_i\mathbf{Z}$ in (A.26), are determined by the same metric structure $\mathbf{g} = \mathbf{f}$.

The main conclusion of this section is that geometrically both (pseudo) Riemannian and (pseudo) Finsler spaces with metric compatible linear connections completely defined by a prescribed metric structure can be modelled equivalently by nonholonomic distributions/ deformations. We can distinguish such spaces only if there are used certain additional physical arguments. For instance, we can consider that a (pseudo) Finsler geometry modelled on a tangent bundle, when v-coordinates y^a are of "velocity" type. This is a very different model from Finsler spacetime models on nonholonomic (pseudo) Riemannian manifolds, when v-coordinates y^a are certain space/time ones but subjected to nonholonomic constraints.

2.2 Einstein equations on (pseudo) Finsler/ nonholonomic spacetimes

For fundamental field interactions, we can distinguish more explicitly the physical models with (pseudo) Finsler spaces from those on holonomic (pseudo) Riemannian manifolds. In general relativity, the Einstein equations are postulated in the form

$$\text{ }_iE_{\alpha\beta} = \varkappa \text{ }_i\Upsilon_{\alpha\beta}, \quad (4)$$

where ${}_1E_{\alpha\beta}$ is the Einstein tensor for the Levi-Civita connection ∇ , \varkappa is the gravitational constant and the energy-momentum tensor ${}_1\Upsilon_{\alpha\beta}$ is constructed for matter fields using (\mathbf{g}, ∇) . Having prescribed a generating function \mathbf{f} , these equations can be written equivalently in Finsler variables, for instance, if $(\mathbf{g}, \nabla) \rightarrow (F : \mathbf{f}, {}^c\mathbf{N}, \hat{\mathbf{D}})$, following the distortion formula (A.26). Nevertheless, the equations (4) are explicitly stated for the Levi-Civita connection ∇ and they are very different from, for instance, the nonholonomic field equations

$$\hat{\mathbf{E}}_{\alpha\beta} = \varkappa \hat{\Upsilon}_{\alpha\beta} \quad (5)$$

constructed for the Einstein d-tensor $\hat{\mathbf{E}}_{\alpha\beta}$ of the canonical d-connection $\hat{\mathbf{D}}$ (see formulas (A.23) for (A.25)) and any general $\hat{\Upsilon}_{\alpha\beta}$ determined by geometric (gravitational field) data $(\mathbf{g}, \hat{\mathbf{D}})$ and matter fields.

We can express the Einstein equations (4) in an equivalent form using canonical Finsler variables,

$$\hat{\mathbf{E}}_{\alpha\beta} = \varkappa {}^c\hat{\Upsilon}_{\alpha\beta}, \quad (6)$$

if the source

$${}^c\hat{\Upsilon}_{\alpha\beta} = {}_1\hat{\Upsilon}_{\alpha\beta} + {}_1\hat{\mathbf{Z}}_{\alpha\beta}$$

is determined by ${}_1\Upsilon_{\alpha\beta}$ rewritten in variables $(\mathbf{g}, \hat{\mathbf{D}})$ and the canonical distortion of the Ricci tensor, ${}_1\hat{\mathbf{Z}}_{\alpha\beta}$, is computed as ${}_1\hat{\mathbf{Z}}_{\beta\gamma} \doteq {}_1\hat{\mathbf{Z}}_{\beta\gamma\alpha}^\alpha$ for $\hat{\mathbf{R}}_{\beta\gamma\delta}^\alpha = {}_1R_{\beta\gamma\delta}^\alpha + {}_1\hat{\mathbf{Z}}_{\beta\gamma\delta}^\alpha$ induced by deformations (A.26).

Even the equations (4) and (6) are equivalent on a nonholonomic (pseudo) Riemannian manifold \mathbf{V} , the last form may contain a more rich geometric and physical information about an induced canonical Finsler structure if we fix in explicit form a generating function $F(u)$. The equations (6) do not brock the general covariance, or local Lorentz invariance, because on \mathbf{V} we can consider any (with nondegenerated Hessian (A.1)) generating Finsler function/ variables. It is a matter of convenience with what type of equations, (4) and (6), we chose to work in Einstein gravity. For instance, in a series of our works [7, 13, 18, 19], we used the variant (6) which was more convenient for constructing exact solutions with generic off-diagonal metrics and nonholonomic constraints (in nonholonomic form, the Einstein equations became exactly integrable).

For models of (pseudo) Finsler geometry on tangent bundles, $\mathbf{V} = TM$, the equations (6) in total space, define a complete model of Finsler gravity with metrics and connections depending additionally on velocities y^a . For instance, in Refs. [49, 50, 51], there were considered Finsler (Lagrange, generalized Lagrange and gauge like) models with Einstein equations of type

(5), which are more general than (6) and with a number of additional constants and Finsler curvature terms. Both types of equations (5) and (6), on TM , are with broken local Lorentz symmetry because prescribing an explicit fundamental Finsler function $F(u)$ we violate both the local symmetry and general covariance and fix an explicit type of fiber frame transform. We consider that equations (6), for models with violated Lorentz symmetry, can be 'more physical' because they are equivalent to the Einstein equations (4) written on tangent bundle but using canonical nonholonomic variables typically used in Lagrange–Finsler geometries and generalizations. The models and analysis of Finsler gravity theories provided in Refs. [49, 50, 51, 55, 53, 48, 47, 46, 45, 44, 42, 41, 38, 37, 36, 34, 33, 32, 31, 30, 21, 20] do not discuss the issue of what type of locally anisotropic field equations should be considered in order to elaborate an integral paradigm both for the general relativity theory and further (non) commutative/supersymmetric/ quantum etc Lagrange–Finsler generalizations. In our opinion, if we accept that one could be realistic some classical or quantum models with (pseudo) Finsler like spacetime on (co) tangent bundles, with broken local Lorentz invariance and even nonmetricity, such theories should be minimally described by field equations of type (6) and their equivalents for the Levi–Civita connection (4), derived on tangent bundles by corresponding nonholonomic deformations and transforms.

Next, we analyze a more particular case of (pseudo) Finsler and non-holonomic Einstein manifolds. In terms of the Levi–Civita connection, the Einstein manifolds are defined by the set of metrics $\mathbf{g} = \{g_{\alpha\beta}\}$ solving the equations

$${}_1R_{\alpha\beta} = \lambda g_{\alpha\beta}, \quad (7)$$

where ${}_1R_{\alpha\beta}$ is the Ricci tensor for ∇ and λ is the cosmological constant. These Einstein equations are just (4) with a particular type of cosmological constant source. For the canonical d-connection $\hat{\mathbf{D}}$, we can consider non-holonomic Einstein spaces defined by d-metrics $\mathbf{g}_{\alpha\beta} = [g_{ij}, h_{ab}]$ as solutions of equations of type

$$\begin{aligned} \hat{R}^i_j &= {}^h\lambda(u)\delta^i_j, \quad \hat{R}^a_b = {}^v\lambda(u)\delta^a_b, \\ \hat{R}_{ia} &= \hat{R}_{ai} = 0 \end{aligned} \quad (8)$$

where $\hat{\mathbf{R}}_{\alpha\beta} = \{\hat{R}_{ij}, \hat{R}_{ia}, \hat{R}_{ai}, \hat{R}_{ab}\}$ are components of the Ricci d-tensor (A.21) computed for coefficients (A.25), ${}^h\lambda(u)$ and ${}^v\lambda(u)$ are some locally anisotropic h- and v-polarizations of the cosmological constant, and δ^i_j , for instance, is the Kronecker symbol. The nonholonomic equations (8)

consist a particular case of (5) and define a class of Finsler–Einstein spaces for $\hat{\mathbf{D}}$.

We have to consider additional nonholonomic constraints on the integral variety (i.e. the space of solutions) of (8) and define solutions of (6) and/or (8). For instance, we analyzed such classes of constraints and solutions in general relativity and noncommutative generalizations [7, 13, 18, 19, 9], see also reviews of exact solutions in [8, 10], and nonholonomic Ricci flow theory and gravity [43, 59, 60, 61, 62, 63]. The idea was to construct certain more general exact solutions for equations with $\hat{\mathbf{D}}$ and $\hat{\mathbf{R}}_{\alpha\beta}$ and then to select some explicit nonholonomic configurations when the distortion tensor vanishes, ${}_N Z^\gamma_{\alpha\beta} = 0$, see formulas (A.27), stating a formal equality of coefficients ${}_N \Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta}$, with respect to a N-adapted frame, even, in general, $\hat{\mathbf{D}} \neq \nabla$. This is possible because of different rules of transforms for linear connections and d-connections, see discussion at the end of Appendix.⁴ As a result, we were able to elaborate a new, very general, geometric method of constructing exact solutions with generic nonholonomic metrics and nontrivial nonholonomic structures in gravity theories (the so-called anholonomic frame method, see reviews of results in Refs. [7, 8, 10]).

Nonholonomic configurations with ${}_N Z^\gamma_{\alpha\beta} = 0$ present a substantial interest because they allow us to model a subclass of (pseudo) Finsler spaces as exact solutions in Einstein gravity. Such Finsler metrics and connections are not restricted by modern experimental data [21, 20] and they do not involve violations of the local Lorentz invariance, or metric noncompatibility of Finsler structures. In next sections, we shall provide explicit examples of (pseudo) Finsler black holes and nonholonomic configurations which can be modelled on Einstein spaces by generic off-diagonal metrics.

⁴We note that the Levi-Civita and the canonical d-connection are with different Riemannian tensors even for certain nonholonomic "degenerated" configurations both their Ricci tensors may be equal, or vanish. Such degenerations are possible when certain conditions are imposed on parameters and nonholonomic frame coefficients in such a way that the necessary sums for contractions of metric coefficients with products of linear connections' distortion tensors vanish for nonholonomic deformations of the Einstein/Ricci tensors. But this results in additional distortions, for instance, of the corresponding Riemannian/Weyl tensors. Here one should be emphasized that the d-connection and N-connection geometries are different from that of usual affine connections and, in particular, of the Levi-Civita connection. Such connections are subjected, in general, to different rules of frame/coordinate transform and their nonholonomic deformations.

3 Off-Diagonal Ansatz and Exact Solutions

We consider an ansatz for d-metric \mathbf{g} (1) when

$$\begin{aligned}\mathbf{g} &= g_i dx^i \otimes dx^i + h_3 \delta v \otimes \delta v + h_4 \delta y \otimes \delta y, \\ \delta v &= dv + w_j dx^j, \quad \delta y = dy + n_j dx^j\end{aligned}\tag{9}$$

is with nontrivial coefficients being functions of necessary smooth class

$$g_i = g_i(x^k), h_a = h_a(x^i, v), w_j = w_j(x^k, v), n_j = n_j(x^k, v),\tag{10}$$

where the N-connection coefficients are $N_{i'}^3 = w_{i'}$, $N_{i'}^4 = n_{i'}$ and coordinates are parametrized in the form $x^{k'} = x^{k'}(x^k)$, $y^{3'} = v$ and $y^{4'} = y$. The above formulas are for a generic off-diagonal metric with 2+2 splitting when the h-metric coefficients depend on two variables and the v-metric and N-connection coefficients depend on three variables (x^i, v) . It has one Killing vector $e_y = \partial/\partial y$ because there is a frame basis when the coefficients do not depend on variable y . How to construct exact solutions for a general ansatz in general relativity and Einstein equations in nonholonomic variables is analyzed in Refs. [7, 18, 19], see also generalizations and reviews of results in [8, 10, 9]. In this work, we shall omit technical details on constructing exact solutions following the anholonomic frame method.

For the ansatz (9), the system of Einstein equations for the canonical d-connection (8) transform into a system of partial differential equations:

$$\begin{aligned}\widehat{R}_1^1 &= \widehat{R}_2^2 \\ &= \frac{1}{2g_1 g_2} \left[\frac{g_1^\bullet g_2^\bullet}{2g_1} + \frac{(g_2^\bullet)^2}{2g_2} - g_2^{\bullet\bullet} + \frac{g_1' g_2'}{2g_2} + \frac{(g_1')^2}{2g_1} - g_1'' \right] = {}^h \lambda(x^i),\end{aligned}\tag{11}$$

$$\widehat{R}_3^3 = \widehat{R}_4^4 = \frac{h_4^*}{2h_3 h_4} \left(\ln \left| \frac{\sqrt{|h_3 h_4|}}{h_4^*} \right| \right)^* = {}^v \lambda(x^i, v),\tag{12}$$

$$\widehat{R}_{3i} = -w_i \frac{\beta}{2h_4} - \frac{\alpha_i}{2h_4} = 0,\tag{13}$$

$$\widehat{R}_{4i} = -\frac{h_4}{2h_3} [n_i^{**} + \gamma n_i^*] = 0,\tag{14}$$

where, for $h_{3,4}^* \neq 0$,

$$\phi = \ln \left| \frac{h_4^*}{\sqrt{|h_3 h_4|}} \right|, \quad \alpha_i = h_4^* \partial_i \phi, \quad \beta = h_4^* \phi^*, \quad \gamma = \left(\ln |h_4|^{3/2} / |h_3| \right)^*,\tag{15}$$

we consider partial derivatives written in the form $a^\bullet = \partial a / \partial x^1$, $a' = \partial a / \partial x^2$, $a^* = \partial a / \partial v$.

The system (11)–(14) can be integrated in very general forms depending on the type of polarizations of the cosmological constant. For simplicity, in this section, we consider two classes of exact solutions of equations (8) when ${}^h\lambda(x^i) = {}^v\lambda(x^i, v) = \lambda = \text{const} \neq 0$ and $\lambda = 0$ (vacuum configurations). We shall also state the nonholonomic conditions ${}_1Z^\gamma_{\alpha\beta} = 0$ (A.27) when the solutions for the canonical d-connection transform into similar ones for the Levi-Civita connection. There will be analyzed the most general type of such solutions with $h_a^* \neq 0$ and certain nontrivial values of w_i and/or n_i . Here we note that some subclasses of solutions with $h_3^* = 0$ and/or $h_4^* = 0$ include those depending on one–two variables considered in general relativity (for instance, the Taub NUT and/or Schwarzschild solutions). We shall not use cosmological and/or Taub NUT metrics in this work (even a similar geometric techniques can be applied for such locally anisotropic models, see some our previous works [64, 65, 60]) but analyze certain limits to the Schwarzschild and/or Schwarzschild–de Sitter spacetimes and their nontrivial generalizations.

3.1 Solutions for nonholonomic Einstein spaces, $\lambda = \text{const}$

A class of exact solutions of (8) with cosmological constant for the ansatz (9) is parametrized by d-metrics of type

$$\begin{aligned} {}^\lambda\mathring{g} &= \epsilon_1 e^{\phi(x^i)} dx^1 \otimes dx^1 + \epsilon_2 e^{\phi(x^i)} dx^2 \otimes dx^2 \\ &\quad + h_3(x^k, v) \delta v \otimes \delta v + h_4(x^k, v) \delta y \otimes \delta y, \\ \delta v &= dv + w_i(x^k, v) dx^i, \quad \delta y = dy + n_i(x^k, v) dx^i, \end{aligned} \quad (16)$$

for any signatures $\epsilon_\alpha = \pm 1$, where the coefficients are any functions satisfying (respectively) the conditions,

$$\begin{aligned} \epsilon_1 \underline{\phi}^{\bullet\bullet}(x^k) + \epsilon_2 \underline{\phi}''(x^k) &= -2\epsilon_1 \epsilon_2 \lambda; \\ h_3 &= \pm \frac{(\phi^*)^2}{4\lambda} e^{-2\phi(x^i)}, \quad h_4 = \mp \frac{1}{4\lambda} e^{2(\phi - \phi(x^i))}; \\ w_i &= -\partial_i \phi / \phi^*; \\ n_i &= {}^1n_i(x^k) + {}^2n_i(x^k) \int (\phi^*)^2 e^{-2(\phi - \phi(x^i))} dv, \\ &= {}^1n_i(x^k) + {}^2n_i(x^k) \int e^{-4\phi} \frac{(h_4^*)^2}{h_4} dv, \quad n_i^* \neq 0; \\ &= {}^1n_i(x^k), \quad n_i^* = 0; \end{aligned} \quad (17)$$

for any nonzero h_a and h_a^* and (integrating) functions ${}^1n_i(x^k)$, ${}^2n_i(x^k)$, generating function $\phi(x^i, v)$ (15), and ${}^0\phi(x^i)$ to be determined from certain boundary conditions for a fixed system of coordinates. There are two classes of solutions (17) constructed for a nontrivial λ . The first one is singular for $\lambda \rightarrow 0$ if we chose a generation function $\phi(x^i, v)$ not depending on λ . It is possible to eliminate such singularities for certain parametric dependencies of type $\phi(\lambda, x^i, v)$, for instance, when such a function is linear on λ .

We have to impose additional constraints on such coefficients in order to satisfy the conditions $Z_{\alpha\beta}^\gamma = 0$ (A.27) and generate solutions of the Einstein equations (7) for the Levi-Civita connection:

$$\begin{aligned} (2e^{2\phi}\phi - \lambda)(\phi^*)^2 &= 0, \phi \neq 0, \phi^* \neq 0; \\ w_1 w_2 \left(\ln \left| \frac{w_1}{w_2} \right| \right)^* &= w_2^\bullet - w_1', w_i^* \neq 0; \\ w_2^\bullet - w_1' &= 0, w_i^* = 0; \\ {}^1n_1'(x^k) - {}^1n_2^\bullet(x^k) &= 0, \end{aligned} \quad (18)$$

which holds for any $\phi(x^i, v) = \text{const}$ if we include configurations with $\phi^* = 0$.

3.2 Solutions for (non) holonomic vacuum spaces, $\lambda = 0$

Because of generic nonlinear off-diagonal and possible nonholonomic character of Einstein equations, the vacuum solutions are generated not just as a simple limit $\lambda \rightarrow 0$ of coefficients (17) and (18). Such a limit to vacuum configurations should be considered for equations (11)–(14) with zero sources on the right part with a further integration on separated variables, see details in [7, 8, 10, 9]. This way we construct a class of vacuum solutions of the Einstein equations for the canonical d-connection, $\hat{\mathbf{R}}_{\alpha\beta} = 0$ in (8) by d-metrics $\hat{\mathbf{g}}$ parametrized in the form (16) but with coefficients satisfying conditions

$$\begin{aligned} \epsilon_1 \underline{\phi}^{\bullet\bullet}(x^k) + \epsilon_2 \underline{\phi}''(x^k) &= 0; \\ h_3 &= \pm e^{-2} {}^0\phi \frac{(h_4^*)^2}{h_4} \text{ for given } h_4(x^i, v), \phi = {}^0\phi = \text{const}; \\ w_i &= w_i(x^i, v), \text{ for any such functions if } \lambda = 0; \\ n_i &= {}^1n_i(x^k) + {}^2n_i(x^k) \int (h_4^*)^2 |h_4|^{-5/2} dv, n_i^* \neq 0; \\ &= {}^1n_i(x^k), n_i^* = 0. \end{aligned} \quad (19)$$

We get vacuum solutions $\hat{\mathbf{g}}$ of the Einstein equations (7) for the Levi-Civita connection, i.e of ${}_1R_{\alpha\beta} = 0$, if we impose additional constraints on

coefficients of d-metric,

$$\begin{aligned}
h_3 &= \pm 4 \left[\left(\sqrt{|h_4|} \right)^* \right]^2, \quad h_4^* \neq 0; \\
w_1 w_2 \left(\ln \left| \frac{w_1}{w_2} \right| \right)^* &= w_2^\bullet - w_1', \quad w_i^* \neq 0; \\
w_2^\bullet - w_1' &= 0, \quad w_i^* = 0; \\
{}^1 n_1'(x^k) - {}^1 n_2^\bullet(x^k) &= 0,
\end{aligned} \tag{20}$$

for $e^{-2 \cdot 0 \phi} = 1$.

It should be emphasized that the bulk of vacuum and cosmological solutions in general relativity outlined in Refs. [2, 4] can be considered as particular cases of metrics with $h_4^* = 0, w_i^* = 0$ and/or $n_i^* = 0$, for corresponding systems of reference. In our approach, we work with more general classes of off-diagonal metrics with certain coefficients depending on three variables. Such solutions in general relativity can be generated if we impose certain nonholonomic constraints on integral varieties of corresponding systems of partial equations. The former analytic and computer numeric programs (for instance, the standard ones with Maple/ Mathematica) for constructing solutions in gravity theories can not be directly applied for alternative verifications of our solutions because those approaches do not encode constraints of type (18) or (20). Nevertheless, it is possible to check in general analytic form, see all details summarized in Part II of [10], that the Einstein cosmological/ vacuum solutions are satisfied for such general off-diagonal ansatz of metrics and various types of d- and N-connections.

3.3 Nonholonomic deformations of the Schwarzschild metric

We consider a diagonal metric

$${}^\varepsilon \mathbf{g} = -d\xi \otimes d\xi - r^2(\xi) d\vartheta \otimes d\vartheta - r^2(\xi) \sin^2 \vartheta d\varphi \otimes d\varphi + \varpi^2(\xi) dt \otimes dt, \tag{21}$$

where the local coordinates and nontrivial metric coefficients are parametrized in the form

$$\begin{aligned}
x^1 &= \xi, x^2 = \vartheta, y^3 = \varphi, y^4 = t, \\
\check{g}_1 &= -1, \check{g}_2 = -r^2(\xi), \check{h}_3 = -r^2(\xi) \sin^2 \vartheta, \check{h}_4 = \varpi^2(\xi),
\end{aligned} \tag{22}$$

for

$$\xi = \int dr \left| 1 - \frac{2\mu_0}{r} + \frac{\varepsilon}{r^2} \right|^{1/2} \text{ and } \varpi^2(r) = 1 - \frac{2\mu_0}{r} + \frac{\varepsilon}{r^2}.$$

For the constants $\varepsilon = 0$ and μ_0 being a point mass, the metric ${}^\varepsilon \mathbf{g}$ (21) is just that for the Schwarzschild solution written in spacetime spherical coordinates $(r, \vartheta, \varphi, t)$.⁵

Let us consider nonholonomic deformations when $g_i = \eta_i \check{g}_i$ and $h_a = \eta_a \check{h}_a$ and w_i, n_i are some nontrivial functions, where $(\check{g}_i, \check{h}_a)$ are given by data (22), to an ansatz

$$\begin{aligned} {}^\varepsilon \eta \mathbf{g} &= -\eta_1(\xi) d\xi \otimes d\xi - \eta_2(\xi) r^2(\xi) d\vartheta \otimes d\vartheta \\ &\quad - \eta_3(\xi, \vartheta, \varphi) r^2(\xi) \sin^2 \vartheta \delta\varphi \otimes \delta\varphi + \eta_4(\xi, \vartheta, \varphi) \varpi^2(\xi) \delta t \otimes \delta t, \\ \delta\varphi &= d\varphi + w_1(\xi, \vartheta, \varphi) d\xi + w_2(\xi, \vartheta, \varphi) d\vartheta, \\ \delta t &= dt + n_1(\xi, \vartheta) d\xi + n_2(\xi, \vartheta) d\vartheta, \end{aligned} \quad (23)$$

for which the coefficients are constrained to define nonholonomic Einstein spaces when the conditions (16) are satisfied. There are used 3-d spacial spherical coordinates $(\xi(r), \vartheta, \varphi)$, or (r, ϑ, φ) , for a class of metrics of type (9) with coefficients of type (10).

The equation (12) for zero source states certain relations between the coefficients of the vertical metric and respective polarization functions,

$$\begin{aligned} h_3 &= -h_0^2 (b^*)^2 = \eta_3(\xi, \vartheta, \varphi) r^2(\xi) \sin^2 \vartheta \\ h_4 &= b^2 = \eta_4(\xi, \vartheta, \varphi) \varpi^2(\xi), \end{aligned} \quad (24)$$

for $|\eta_3| = (h_0)^2 |\check{h}_4 / \check{h}_3| \left[\left(\sqrt{|\eta_4|} \right)^* \right]^2$. In these formulas, we have to chose $h_0 = \text{const}$ (it must be $h_0 = 2$ in order to satisfy the first condition (20)), where \check{h}_a are stated by the Schwarzschild solution for the chosen system of coordinates and η_4 can be any function satisfying the condition $\eta_4^* \neq 0$. We generate a class of solutions for any function $b(\xi, \vartheta, \varphi)$ with $b^* \neq 0$. For different purposes, it is more convenient to work directly with η_4 , for $\eta_4^* \neq 0$, and/or h_4 , for $h_4^* \neq 0$.

It is possible to compute the polarizations η_1 and η_2 , when $\eta_1 = \eta_2 r^2 = e^{\psi(\xi, \vartheta)}$, from (11) with zero source, written in the form

$$\psi^{\bullet\bullet} + \psi'' = 0.$$

Putting the defined values of the coefficients in the ansatz (23), we find a class of exact vacuum solutions of the Einstein equations defining stationary

⁵For simplicity, in this work, we shall consider only the case of "pure" gravitational vacuum solutions, not analyzing a more general possibility when $\varepsilon = e^2$ can be related to the electric charge for the Reissner–Nordström metric for the so-called holonomic electro vacuum configurations (see details, for example, [5]). We treat ε as a small parameter (eccentricity) defining a small deformation of a circle into an ellipse.

nonholonomic deformations of the Schwarzschild metric,

$$\begin{aligned}
{}^\varepsilon \mathbf{g} &= -e^\psi (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta) \\
&\quad -4 \left[\left(\sqrt{|\eta_4|} \right)^* \right]^2 \varpi^2 \delta\varphi \otimes \delta\varphi + \eta_4 \varpi^2 \delta t \otimes \delta t, \\
\delta\varphi &= d\varphi + w_1(\xi, \vartheta, \varphi) d\xi + w_2(\xi, \vartheta, \varphi) d\vartheta, \\
\delta t &= dt + {}^1n_1 d\xi + {}^1n_2 d\vartheta.
\end{aligned} \tag{25}$$

The N-connection coefficients w_i and 1n_i must satisfy the conditions (20) in order to get vacuum metrics in Einstein gravity. Such vacuum solutions are for nonholonomic deformations of a static black hole metric into (non) holonomic Einstein spaces with locally anisotropic backgrounds (on coordinate φ) defined by an arbitrary function $\eta_4(\xi, \vartheta, \varphi)$ with $\partial_\varphi \eta_4 \neq 0$, an arbitrary $\psi(\xi, \vartheta)$ solving the 2-d Laplace equation and certain integration functions ${}^1w_i(\xi, \vartheta, \varphi)$ and ${}^1n_i(\xi, \vartheta)$. In general, the solutions from the target set of metrics do not define black holes and do not describe obvious physical situations. Nevertheless, they preserve the singular character of the coefficient ϖ^2 vanishing on the horizon of a Schwarzschild black hole if we take only smooth integration functions for some small deformation parameters ε . We can also consider a prescribed physical situation when, for instance, η_4 mimics 3-d, or 2-d, solitonic polarizations on coordinates ξ, ϑ, φ , or on ξ, φ .

3.4 Solutions with linear parametric nonholonomic polarizations

From a very general d-metric (25) defining nonholonomic deformations of the Schwarzschild solution depending on parameter ε , we select locally anisotropic configurations with possible physical interpretation of gravitational vacuum configurations with spherical and/or rotoid (ellipsoid) symmetry. Let us consider a generating function of type

$$b^2 = q(\xi, \vartheta, \varphi) + \varepsilon s(\xi, \vartheta, \varphi) \tag{26}$$

and, for simplicity, restrict our analysis only with linear decompositions on a small parameter ε , with $0 < \varepsilon \ll 1$. This way, we shall construct exact solutions with off-diagonal metrics of the Einstein equations depending on ε which for rotoid configurations can be considered as a small eccentricity. From a formal point of view, we can summarize on all orders $\varepsilon^2, \varepsilon^3 \dots$ stating such recurrent formulas for coefficients when get convergent series to some functions depending both on spacetime coordinates and a parameter ε , see a detailed analysis in Ref. [7].

A straightforward computation with (26) allows us to write

$$(b^*)^2 = \left[(\sqrt{|q|})^* \right]^2 \left[1 + \varepsilon \frac{1}{(\sqrt{|q|})^*} \left(\frac{s}{\sqrt{|q|}} \right)^* \right]$$

and compute the vertical coefficients of d-metric (25), i.e h_3 and h_4 (and corresponding polarizations η_3 and η_4) using formulas (24).

In a particular case, we can generate nonholonomic deformations of the Schwarzschild solution not depending on ε if we consider $\varepsilon = 0$ in the above formulas consider only nonholonomic deformations with $b^2 = q$ and $(b^*)^2 = \left[(\sqrt{|q|})^* \right]^2$.

Nonholonomic deformations to rotoid configurations are possible if we chose

$$q = 1 - \frac{2\mu(\xi, \vartheta, \varphi)}{r} \text{ and } s = \frac{q_0(r)}{4\mu^2} \sin(\omega_0\varphi + \varphi_0), \quad (27)$$

for $\mu(\xi, \vartheta, \varphi) = \mu_0 + \varepsilon\mu_1(\xi, \vartheta, \varphi)$ (locally anisotropically polarized mass) with certain constants μ, ω_0 and φ_0 and arbitrary functions/polarizations $\mu_1(\xi, \vartheta, \varphi)$ and $q_0(r)$ to be determined from some boundary conditions, with ε being the eccentricity.⁶ The possibility to treat ε as an eccentricity follows from the condition that the coefficient $h_4 = b^2 = \eta_4(\xi, \vartheta, \varphi)\varpi^2(\xi)$ becomes zero for data (27) if

$$r_+ \simeq \frac{2\mu_0}{1 + \varepsilon \frac{q_0(r)}{4\mu^2} \sin(\omega_0\varphi + \varphi_0)}.$$

This condition defines a small deformation of the Schwarzschild spherical horizon into an ellipsoidal one (rotoid configuration with eccentricity ε).

⁶A nonholonomic Einstein vacuum is modelled as a continuum media with possible singularities which because of generic nonlinear character of gravitational interactions may result in effective locally anisotropic polarizations of fundamental physical constants. Similar effects of polarization of constants can be measured experimentally in classical and nonlinear electrodynamics, for instance, in various types of continuous media and dislocations and disclinations. For spherical symmetries, with local isotropy, the point mass is approximated by a constant μ_0 . We have to consider anisotropically polarized masses of type $\mu_1(\xi, \varphi, \vartheta)$ for locally anisotropic models (for instance, with ellipsoidal symmetry) in general relativity and various types of gravity theories. Such sources should be introduced following certain phenomenological arguments, like in classical electrodynamics, or computed as certain quasi-classical approximations from a quantum gravity model, like in quantum electrodynamics.

Let us summarize the coefficients of a d-metric defining rotoid type solutions:

$$\begin{aligned}
{}^{rot}\mathbf{g} &= -e^\psi (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta) \\
&\quad -4 \left[(\sqrt{|q|})^* \right]^2 \left[1 + \varepsilon \frac{1}{(\sqrt{|q|})^*} \left(\frac{s}{\sqrt{|q|}} \right)^* \right] \delta\varphi \otimes \delta\varphi \\
&\quad + (q + \varepsilon s) \delta t \otimes \delta t, \\
\delta\varphi &= d\varphi + w_1 d\xi + w_2 d\vartheta, \quad \delta t = dt + {}^1n_1 d\xi + {}^1n_2 d\vartheta,
\end{aligned} \tag{28}$$

with functions $q(\xi, \vartheta, \varphi)$ and $s(\xi, \vartheta, \varphi)$ given by formulas (27) and N-connection coefficients $w_i(\xi, \vartheta, \varphi)$ and $n_i = {}^1n_i(\xi, \vartheta)$ subjected to conditions of type (20),

$$\begin{aligned}
w_1 w_2 \left(\ln \left| \frac{w_1}{w_2} \right| \right)^* &= w_2^\bullet - w_1', \quad w_i^* \neq 0; \\
\text{or } w_2^\bullet - w_1' &= 0, \quad w_i^* = 0; \quad {}^1n_1'(\xi, \vartheta) - {}^1n_2^\bullet(\xi, \vartheta) = 0
\end{aligned}$$

and $\psi(\xi, \vartheta)$ being any function for which $\psi^{\bullet\bullet} + \psi'' = 0$.

Finally we emphasize that the d-metrics with rotoid symmetry constructed in this section are different from those considered in our previous works [18, 19, 9]. In general, they do not define black hole solutions. Nevertheless, for small eccentricities, we get stationary configurations for the so-called black ellipsoid solutions (their stability and properties can be analyzed following the methods elaborated in the mentioned works, see also a summary of results and generalizations for various types of locally anisotropic gravity models in Ref. [10]).

4 Finsler Black Holes, Ellipsoids and Nonlinear Gravitational Waves

The next step to be taken is to show how we can construct black hole solutions in a (pseudo) Finsler spacetime using certain analogy with the Schwarzschild solution rewritten in Finsler variables. There are several avenues to be explored, and we separate the material into three subsections. The first one is for nonholonomic rotoid deformations of Einstein metrics when the resulting general off-diagonal metrics contain a nontrivial cosmological constant. The second one concerns embedding of black hole solutions and their nonholonomic deformations into nontrivial backgrounds of nonlinear waves. Finally, the third subsection is devoted to Finsler variables

in general relativity and analogs of the Schwarzschild solution in (pseudo) Finsler spacetimes.

4.1 Nonholonomic rotoid deformations of Einstein metrics

Using the anholonomic frame method we can construct a class of solutions with nontrivial cosmological constant possessing different limits, for large radial distances and small nonholonomic deformations, than vacuum configurations considered in section 3.4. Such stationary metrics belong to the class of d-metrics (16) defining exact solutions of gravitational equations (8).

Let us consider a diagonal metric of type

$${}^\varepsilon \mathbf{g} = -d\xi \otimes d\xi - r^2(\xi) d\vartheta \otimes d\vartheta - r^2(\xi) \sin^2 \vartheta d\varphi \otimes d\varphi + {}_\lambda \varpi^2(\xi) dt \otimes dt, \quad (29)$$

where the local coordinates and nontrivial metric coefficients are parametrized in the form

$$\begin{aligned} x^1 &= \xi, x^2 = \vartheta, y^3 = \varphi, y^4 = t, \\ \check{g}_1 &= -1, \check{g}_2 = -r^2(\xi), \check{h}_3 = -r^2(\xi) \sin^2 \vartheta, \check{h}_4 = {}_\lambda \varpi^2(\xi) \end{aligned}$$

for

$$\begin{aligned} \xi &= \int dr \left| 1 - \frac{2\mu}{r} + \varepsilon \left(\frac{1}{r^2} + \frac{\lambda}{3} {}_4\kappa^2 r^2 \right) \right|^{1/2}, \\ {}_\lambda \varpi^2(r) &= 1 - \frac{2\mu}{r} + \varepsilon \left(\frac{1}{r^2} - \frac{\lambda}{3} {}_4\kappa^2 r^2 \right), \end{aligned}$$

where ${}_4\kappa^2 = 1/M_*^2$ stands for the 4-dimensional Newton's constant, $\lambda = \varepsilon \underline{\lambda}$ is a positive cosmological constant and μ_1 is the so-called ADM mass, see Ref. [66] for a review of results on Schwarzschild-de Sitter black holes in $(4 + n_1)$ -dimensions, for $n_1 = 1, 2, \dots$. For the constants $\varepsilon \rightarrow 0$ and $\underline{\mu}$ taken to be a point mass (in general, for a stationary locally anisotropic model this is a function of type $\underline{\mu} = \underline{\mu}_0 + \varepsilon \underline{\mu}_1(\xi, \vartheta, \varphi)$, for $\underline{\mu}_0 = \text{const}$ and function $\underline{\mu}_1(\xi, \vartheta, \varphi)$ taken from phenomenological considerations), the metric ${}^\varepsilon \mathbf{g}$ (29) has a true singularity at $r = 0$ and the equation

$$1 - \frac{2\underline{\mu}_0}{r} + \frac{1}{3} \lambda {}_4\kappa^2 r^2 = 0$$

has three solutions for not small r (when we can neglect the term $1/r^2$) corresponding to three horizons for this spacetime. There are only two

real positive roots because of positivity of radial coordinate r : the first one corresponds to the so-called "cosmological horizon" and the second one (the smaller) is for the "black hole event horizon". A nontrivial parameter ε deforms the metric black hole metric nonholonomically into a d-metric which (in general) does not satisfy the Einstein equations. We have to introduce additional off-diagonal terms and new nonholonomic constraints in order to define nonholonomic transforms into an exact solution.

We chose local coordinates from (22) and consider the ansatz

$$\begin{aligned}\lambda \underline{\mathbf{g}} &= -e^{\underline{\phi}(\xi, \vartheta)} d\xi \otimes d\xi - e^{\underline{\phi}(\xi, \vartheta)} d\vartheta \otimes d\vartheta \\ &\quad + h_3(\xi, \vartheta, \varphi) \delta\varphi \otimes \delta\varphi + h_4(\xi, \vartheta, \varphi) \delta t \otimes \delta t, \\ \delta\varphi &= d\varphi + w_1(\xi, \vartheta, \varphi) d\xi + w_2(\xi, \vartheta, \varphi) d\vartheta, \\ \delta t &= dt + n_1(\xi, \vartheta, \varphi) d\xi + n_2(\xi, \vartheta, \varphi) d\vartheta,\end{aligned}$$

for $h_3 = -h_0^2(b^*)^2 = \eta_3(\xi, \vartheta, \varphi)r^2(\xi)\sin^2\vartheta$, $h_4 = b^2 = \eta_4(\xi, \vartheta, \varphi)\lambda\varpi^2(\xi)$, where the coefficients satisfy the conditions,

$$\begin{aligned}\underline{\phi}^{\bullet\bullet}(\xi, \vartheta) + \underline{\phi}''(\xi, \vartheta) &= 2\lambda; \\ h_3 &= \pm \frac{(\phi^*)^2}{4\lambda} e^{-2\phi(\xi, \vartheta)}, \quad h_4 = \mp \frac{1}{4\lambda} e^{2(\phi - \phi(\xi, \vartheta))}; \\ w_i &= -\partial_i \phi / \phi^*; \\ n_i &= {}^1n_i(\xi, \vartheta) + {}^2n_i(\xi, \vartheta) \int (\phi^*)^2 e^{-2(\phi - \phi(\xi, \vartheta))} d\varphi, \\ &= {}^1n_i(\xi, \vartheta) + {}^2n_i(\xi, \vartheta) \int e^{-4\phi} \frac{(h_4^*)^2}{h_4} d\varphi, \quad n_i^* \neq 0; \\ &= {}^1n_i(\xi, \vartheta), \quad n_i^* = 0;\end{aligned}\tag{30}$$

for any nonzero h_a and h_a^* and (integrating) functions ${}^1n_i(\xi, \vartheta)$, ${}^2n_i(\xi, \vartheta)$, generating function $\phi(\xi, \vartheta, \varphi)$ (15), and $\phi(\xi, \vartheta)$ to be determined from certain boundary conditions for a fixed system of coordinates.

In explicit form, the d-metric determining nonholonomic ellipsoid de Sitter configurations is written

$$\begin{aligned}{}^{rot}_\lambda \underline{\mathbf{g}} &= -e^{\underline{\phi}(\xi, \vartheta)} (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta) \\ &\quad - h_0^2 \left[(\sqrt{|q|})^* \right]^2 \left[1 + \varepsilon \frac{1}{(\sqrt{|q|})^*} \left(\frac{\underline{s}}{\sqrt{|q|}} \right)^* \right] \delta\varphi \otimes \delta\varphi \\ &\quad + (\underline{q} + \varepsilon \underline{s}) \delta t \otimes \delta t, \\ \delta\varphi &= d\varphi + w_1 d\xi + w_2 d\vartheta, \quad \delta t = dt + n_1 d\xi + n_2 d\vartheta.\end{aligned}\tag{31}$$

where

$$\underline{q} = 1 - \frac{2 \, {}^1\mu(r, \vartheta, \varphi)}{r}, \quad \underline{s} = \frac{\underline{q}_0(r)}{4\underline{\mu}_0^2} \sin(\omega_0 \varphi + \varphi_0),$$

with ${}^1\mu(r, \vartheta, \varphi) = \underline{\mu} + \varepsilon (r^{-2} - \underline{\lambda} \, {}_4\kappa^2 \, r^2/3)/2$, chosen to generate an anisotropic rotoid configuration for the smaller "horizon" (when $h_4 = 0$),

$$r_+ \simeq \frac{2 \, {}^1\mu}{1 + \varepsilon \frac{\underline{q}_0(r)}{4\underline{\mu}_0^2} \sin(\omega_0 \varphi + \varphi_0)},$$

for a corresponding $\underline{q}_0(r)$. The d-metric (31) and N-connection coefficients (30) determines a solution of nonholonomic Einstein equations (8). It is not a solution in general relativity but can be considered in (pseudo) Finsler models of gravity.

We have to impose the condition that the coefficients of the above d-metric satisfy the constraints (18) in order to generate solutions of the Einstein equations (7) for the Levi-Civita connection. From the first constraint, for $\phi^* \neq 0$, we obtain the condition that $\phi(r, \varphi, \vartheta) = \ln |h_4^*/\sqrt{|h_3 h_4|}|$ must be any function defined in non-explicit form from equation $2e^{2\phi}\phi = \lambda$. The set of constraints for the N-connection coefficients is to be satisfied if the integration functions in (30) are chosen in a form when $w_1 w_2 \left(\ln \left| \frac{w_1}{w_2} \right| \right)^* = w_2^\bullet - w_1'$ for $w_i^* \neq 0$; $w_2^\bullet - w_1' = 0$ for $w_i^* = 0$; and take $n_i = {}^1n_i(x^k)$ for ${}^1n_1'(x^k) - {}^1n_2^\bullet(x^k) = 0$.

In the limit $\varepsilon \rightarrow 0$, we get a subclass of solutions of type (31) possessing spherical symmetry but with generic off-diagonal coefficients induced by the N-connection coefficients and depending on cosmological constant. In order to extract from such configurations the Schwarzschild solution, we must select a set of functions with the properties $\phi \rightarrow const, w_i \rightarrow 0, n_i \rightarrow 0$ and $h_4 \rightarrow \varpi^2$. In general, the parametric dependence on cosmological constant, for nonholonomic configurations, is not smooth.

4.2 Rotoids and solitonic distributions

On a N-anholonomic spacetime \mathbf{V} defined by a rotoid d-metric ${}^{rot}\mathbf{g}$ (28), we can consider a static three dimensional solitonic distribution $\eta(\xi, \vartheta, \varphi)$ as a solution of solitonic equation⁷

$$\eta^{\bullet\bullet} + \epsilon(\eta' + 6\eta \, \eta^* + \eta^{***})^* = 0, \quad \epsilon = \pm 1.$$

⁷as a matter of principle, we can consider that η is a solution of any three dimensional solitonic and/ or other nonlinear wave equations

It is possible to define a nonholonomic transform from ${}^{rot}_{st}\mathbf{g}$ to a d-metric ${}^{rot}_{st}\mathbf{g}$ determining a stationary metric for a rotoid in solitonic background in general relativity:

$$\begin{aligned} {}^{rot}_{st}\mathbf{g} &= -e^\psi (d\xi \otimes d\xi + d\vartheta \otimes d\vartheta) \\ &\quad -4 \left[(\sqrt{|\eta q|})^* \right]^2 \left[1 + \varepsilon \frac{1}{(\sqrt{|\eta q|})^*} \left(\frac{s}{\sqrt{|\eta q|}} \right)^* \right] \delta\varphi \otimes \delta\varphi \\ &\quad + \eta (q + \varepsilon s) \delta t \otimes \delta t, \\ \delta\varphi &= d\varphi + w_1 d\xi + w_2 d\vartheta, \quad \delta t = dt + {}^1n_1 d\xi + {}^1n_2 d\vartheta, \end{aligned} \quad (32)$$

where the N-connection coefficients are taken the same as for (28). In the limit $\varepsilon \rightarrow 0$, this metric defines a nonholonomic embedding of the Schwarzschild solution into a solitonic vacuum, which results in a vacuum solution of the Einstein gravity defined by a stationary generic off-diagonal metric. For small polarizations, when $|\eta| \sim 1$, it is preserved the black hole character of metric and the solitonic distribution can be considered as on a Schwarzschild background. It is also possible to take such parameters of η when a black hole is nonholonomically placed on a "gravitational hill" defined by a soliton. All such solutions are stationary; to construct solutions for nonholonomic propagation of black holes in extra dimension and/or as Ricci flows is also possible, see details in Refs. [17, 61, 62] and reviews of results, with solutions for the metric-affine gravity, noncommutative generalizations etc, in [8, 10].

A d-metric (32) can be generalized for (pseudo) Finsler spaces with canonical d-connection as a solution of equations $\widehat{\mathbf{R}}_{\alpha\beta} = 0$ (8) by d-metrics parametrized in the form (16) with stationary coefficients subjected to conditions

$$\begin{aligned} \psi^{\bullet\bullet}(\xi, \vartheta) + \psi''(\xi, \vartheta) &= 0; \\ h_3 &= \pm e^{-2\phi} \frac{(h_4^*)^2}{h_4} \text{ for given } h_4(\xi, \vartheta, \varphi), \quad \phi = {}^0\phi = const; \\ w_i &= w_i(\xi, \vartheta, \varphi) \text{ are any functions if } \lambda = 0; \\ n_i &= {}^1n_i(\xi, \vartheta) + {}^2n_i(\xi, \vartheta) \int (h_4^*)^2 |h_4|^{-5/2} dv, \quad n_i^* \neq 0; \\ &= {}^1n_i(\xi, \vartheta), n_i^* = 0, \end{aligned} \quad (33)$$

for $h_4 = \eta(\xi, \vartheta, \varphi) [q(\xi, \vartheta, \varphi) + \varepsilon s(\xi, \vartheta, \varphi)]$. In the limit $\varepsilon \rightarrow 0$, we get a so-called Schwarzschild black hole solution mapped nonholonomically on a N-anholonomic (pseudo) Riemannian spacetime, or on a nonholonomic tangent bundle.

We get a model of Finsler gravity on a tangent bundle TM with a two-dimensional base M and typical two-dimensional fiber endowed with a pseudo-Euclidean metric when $y^3 = v = \varphi$ is the anisotropic coordinate and $y^4 = t$ is the time like coordinates. Such an exact solution for the Einstein equations for the canonical d-connection is described by a d-metric (32) with coefficients of type (33).

4.3 Nonholonomic transforms, Finsler variables and exact solutions in (pseudo) Finsler gravity theories

Finsler variables can be considered on any (pseudo) Riemannian manifold/ tangent bundle \mathbf{V} if we prescribe a generating fundamental Finsler function $F(x, y)$. This function induces canonical (Finsler) N- and d-connection structures, a class of N-adapted frames and a Sasaky type d-metric \mathbf{f} . By nonholonomic deforms, using corresponding vierbein coefficients, such values can be related to an arbitrary d-metric structure \mathbf{g} on \mathbf{V} , in particular, to an exact solution $\mathring{\mathbf{g}}$.⁸

Let us consider three sets of data:

- a) The values $\mathbf{f}_{\alpha\beta} = [f_{ij}, f_{ab}, {}^cN_i^a]$ (A.10) for

$${}^c\mathbf{e}^\alpha = {}^c\mathbf{e}_{\underline{\alpha}}^\alpha e^\alpha = (e^i = dx^i, {}^c\mathbf{e}^a = dy^a + {}^cN_i^a dx^i),$$

with ${}^c\mathbf{e}_{\underline{\alpha}}^\alpha = [{}^c\mathbf{e}_{\underline{i}}^i = \delta_{\underline{i}}^i, {}^c\mathbf{e}_{\underline{a}}^a]$, $e^\alpha = [dx^i, dy^a]$, defines a (pseudo) Finsler space with canonical N-connection ${}^cN_i^a$. We shall use the canonical d-connection $\hat{\Gamma}_{\alpha\beta}^\gamma$ (A.25) computed for values $\mathbf{f}_{\alpha\beta}$.

- b) The values $\mathbf{g}_{\alpha'\beta'} = [g_{i'j'}, h_{a'b'}, N_{i'}^{a'}]$ (1) for

$$\mathbf{e}^{\alpha'} = \mathbf{e}_{\underline{\alpha'}}^{\alpha'} e^{\alpha'} = (e^{i'} = dx^{i'}, \mathbf{e}^{a'} = dy^{a'} + N_{i'}^{a'} dx^{i'}),$$

⁸In this work, the term exact solution refers to the Einstein equations for the canonical d-connection $\hat{\mathbf{D}}$ and/or the Levi-Civita connection $\hat{\nabla}$. Usually, we state exactly what kind of linear connections are used. In general, we can nonholonomically transform a Finsler d-metric \mathbf{f} into a d-metric \mathbf{g} , when at least one of such d-metrics is not a solution of any gravitational field equations for the corresponding d-connections, and finally to deform such a sequence of two transforms into an exact solution $\mathring{\mathbf{g}}$ but the this does not mean that all corresponding Ricci d-tensors, for instance, will vanish in the case that one of such d-metric is a vacuum solution. If we state for an N-adapted frame structure that $\mathbf{f} = \mathbf{g} = \mathring{\mathbf{g}}$, we argue that we introduce certain \mathbf{f} - and/or $\mathring{\mathbf{g}}$ -variables for a given (pseudo) Riemannian metric \mathbf{g} and this will result in the corresponding equalities of all Ricci/Einstein d-tensors.

with $\mathbf{e}_{\underline{\alpha}}^{\alpha'} = [\mathbf{e}_{\underline{i}}^{i'} = \delta_{\underline{i}}^{i'}, \mathbf{e}_{\underline{a}}^{a'}]$, determines a general (pseudo) Riemannian metric (for purposes of this work, the coefficients $N_{i'}^{a'}$ will be not general ones but taken to satisfy some conditions of type $N_{i'}^{a'} = \mathring{N}_i^a$).

c) The coefficients $\mathring{\mathbf{g}}_{\alpha''\beta''} = [\mathring{g}_{i''j''}, \mathring{h}_{a''b''}, \mathring{N}_{i''}^{a''}]$ for

$$\mathring{\mathbf{e}}^{\alpha''} = \mathring{\mathbf{e}}_{\underline{\alpha}}^{\alpha''} e^{\underline{\alpha}} = (e^{i''} = dx^{i''}, \mathbf{e}^{a''} = dy^{a''} + \mathring{N}_{i''}^{a''} dx^{i''}),$$

with $\mathbf{e}_{\underline{\alpha}}^{\alpha''} = [\mathbf{e}_{\underline{i}}^{i''} = \delta_{\underline{i}}^{i''}, \mathbf{e}_{\underline{a}}^{a''}]$, define a solution of nonholonomic Einstein equations for the canonical d-connection, or its restriction to the case of the Levi-Civita connection; for various classes of (non) holonomic Einstein spaces we can chose $\mathring{\mathbf{g}}_{\alpha''\beta''}$ to be defined by a d-metric (16) with any subsets of coefficients subjected to respective conditions (17), or (18), for a nontrivial cosmological constant, and (19), or (20), for vacuum configurations.

To model a (pseudo) Finsler geometry in general relativity we have to impose the conditions $\mathbf{f} = \mathbf{g} = \mathring{\mathbf{g}}$.

4.3.1 (Pseudo) Riemannian metrics in Finsler variables

By frame transforms any data of type a) can be equivalently expressed as data of type b) and inversely. For $\mathbf{f}_{\alpha\beta} = \mathbf{e}_{\alpha}^{\alpha'} \mathbf{e}_{\beta}^{\beta'} \mathbf{g}_{\alpha'\beta'}$, we write explicit parametrizations

$$f_{ij} = e_{i'}^{i'} e_{j'}^{j'} g_{i'j'} \text{ and } f_{ab} = e_a^{a'} e_b^{b'} g_{a'b'}, \quad (34)$$

$$N_{i'}^{a'} = e_{i'}^i e_a^{a'} {}^c N_i^a, \text{ or } {}^c N_i^a = e_i^{i'} e_{a'}^a N_{i'}^{a'}, \quad (35)$$

with matrices $e_{i'}^{i'} = \begin{pmatrix} e_{1'}^{1'} & e_{2'}^{1'} \\ e_{1'}^{2'} & e_{2'}^{2'} \end{pmatrix}$ and $e_a^{a'} = \begin{pmatrix} e_3^{3'} & e_4^{3'} \\ e_3^{4'} & e_4^{4'} \end{pmatrix}$ were, for instance, $e_{a'}^a$ is inverse to $e_a^{a'}$.

For simplicity, we chose $g_{i'j'} = \text{diag}[g_{1'}, g_{2'}]$, $h_{a'b'} = \text{diag}[h_{3'}, h_{4'}]$ and $N_{i'}^{a'} = (N_{i'}^{3'} = w_{i'}, N_{i'}^{4'} = n_{i'})$. The (pseudo) Finsler data f_{ij} , f_{ab} and ${}^c N_i^a = ({}^c N_i^3 = {}^c w_i, {}^c N_i^4 = {}^c n_i)$ are with diagonal matrices, $f_{ij} = \text{diag}[f_1, f_2]$ and $f_{ab} = \text{diag}[f_3, f_4]$, if the generating function is of type $F = {}^3F(x^1, x^2, v) + {}^4F(x^1, x^2, y)$ for some homogeneous (respectively, on $y^3 = v$ and $y^4 = y$) functions 3F and 4F .⁹ For a diagonal representation with $e_4^{3'} = e_3^{4'} = e_1^{2'} =$

⁹Of course, we can work with arbitrary generating functions $F(x^1, x^2, v, y)$ but this will result in off-diagonal (pseudo) Finsler metrics in N-adapted bases, which would request a more cumbersome matrix calculus.

$e_2^{1'} = 0$, we satisfy the conditions (34) if

$$e_1^{1'} = \pm \sqrt{\left| \frac{f_1}{g_{1'}} \right|}, e_2^{2'} = \pm \sqrt{\left| \frac{f_2}{g_{2'}} \right|}, e_3^{3'} = \pm \sqrt{\left| \frac{f_3}{h_{3'}} \right|}, e_4^{4'} = \pm \sqrt{\left| \frac{f_4}{h_{4'}} \right|}. \quad (36)$$

For any fixed values f_i , f_a and ${}^c w_i, {}^c n_i$ and given $g_{i'}$ and $h_{a'}$, we can compute $w_{i'}$ and $n_{i'}$ as

$$\begin{aligned} w_{1'} &= \pm \sqrt{\left| \frac{g_{1'} f_3}{h_{3'} f_1} \right|} {}^c w_1, w_{2'} = \pm \sqrt{\left| \frac{g_{2'} f_3}{h_{3'} f_2} \right|} {}^c w_2, \\ n_{1'} &= \pm \sqrt{\left| \frac{g_{1'} f_4}{h_{4'} f_1} \right|} {}^c n_1, n_{2'} = \pm \sqrt{\left| \frac{g_{2'} f_4}{h_{4'} f_2} \right|} {}^c n_2 \end{aligned} \quad (37)$$

solving the equations (35).

4.3.2 Anti-diagonal frame transforms and exact solutions

It is also possible to define frame transforms relating data of type b) to some data of type c) and inversely. In this case, the vierbein matrices should be taken to be anti-diagonal in order to keep in mind the possibility to relate data c) with some (pseudo) Finsler ones of type a).

Let us consider

$$\mathring{\mathbf{g}}_{\alpha''\beta''} = \mathring{\mathbf{e}}_{\alpha'}^{\alpha''} \mathring{\mathbf{e}}_{\beta''}^{\beta'} \mathbf{g}_{\alpha'\beta'} \quad (38)$$

parametrized in the form

$$\mathring{g}_{i''j''} = g_{i'j'} \mathring{\mathbf{e}}_{i''}^{i'} \mathring{\mathbf{e}}_{j''}^{j'} + h_{a'b'} \mathring{\mathbf{e}}_{i''}^{a'} \mathring{\mathbf{e}}_{j''}^{b'}, \mathring{h}_{a''b''} = g_{i'j'} \mathring{\mathbf{e}}_{a''}^{i'} \mathring{\mathbf{e}}_{b''}^{j'} + h_{a'b'} \mathring{\mathbf{e}}_{a''}^{a'} \mathring{\mathbf{e}}_{b''}^{b'},$$

for an exact solution of Einstein equations determined by data $\mathring{\mathbf{g}}_{\alpha\beta} = [\mathring{g}_i, \mathring{h}_a, \mathring{N}_i^a]$ in a N-elongated base $\mathring{\mathbf{e}}^\alpha = (dx^i, \mathring{\mathbf{e}}^a = dy^a + \mathring{N}_i^a dx^i)$. For $\mathring{\mathbf{e}}_{i''}^{i'} = \delta_{i''}^{i'}$, $\mathring{\mathbf{e}}_{a''}^{a'} = \delta_{a''}^{a'}$, we write (38) as

$$\mathring{g}_{i''} = g_{i''} + h_{a'} \left(\mathring{\mathbf{e}}_{i''}^{a'} \right)^2, \mathring{h}_{a''} = g_{i'} \left(\mathring{\mathbf{e}}_{a''}^{i'} \right)^2 + h_{a''}, \quad (39)$$

i.e. four equations for eight unknown variables $\mathring{\mathbf{e}}_{i''}^{a'}$ and $\mathring{\mathbf{e}}_{a''}^{i'}$, and

$$\mathring{N}_{i''}^{a''} = \mathring{\mathbf{e}}_{i''}^{i'} \mathring{\mathbf{e}}_{a''}^{a'} \mathring{N}_{i'}^{a'} = \mathring{N}_{i''}^{a''}.$$

For instance, we can solve the algebraic system (39) as

$$\begin{aligned} \mathring{\mathbf{e}}_{1''}^{3'} &= \pm \sqrt{|(\mathring{g}_{1''} - g_{1''})/h_{3'}|}, \mathring{\mathbf{e}}_{2''}^{3'} = 0, \mathring{\mathbf{e}}_{i''}^{4'} = 0, \\ \mathring{\mathbf{e}}_{a''}^{1'} &= 0, \mathring{\mathbf{e}}_{3''}^{2'} = 0, \mathring{\mathbf{e}}_{4''}^{2'} = \pm \sqrt{|(\mathring{h}_{4''} - h_{4''})/g_{2'}|}, \end{aligned}$$

for certain nontrivial/nondegenerate values of metric coefficients.

Using (37), with $\mathring{N}_{i''}^{a''} = N_{i''}^{a'''}$, we get

$$\begin{aligned}\mathring{w}_{1'} &= \pm \sqrt{\left| \frac{g_{1'} f_3}{h_{3'} f_1} \right|} {}^c w_1, \quad \mathring{w}_{2'} = \pm \sqrt{\left| \frac{g_{2'} f_3}{h_{3'} f_2} \right|} {}^c w_2, \\ \mathring{n}_{1'} &= \pm \sqrt{\left| \frac{g_{1'} f_4}{h_{4'} f_1} \right|} {}^c n_1, \quad \mathring{n}_{2'} = \pm \sqrt{\left| \frac{g_{2'} f_4}{h_{4'} f_2} \right|} {}^c n_2.\end{aligned}\tag{40}$$

From these formulas, we compute $g_{i'}, h_{a'}$, when $g_{i'} = \delta_{i'}^{i''} g_{i''}$, $h_{a'} = \delta_{a'}^{a''} h_{a''}$, $\mathring{w}_{i'} = \delta_{i'}^{i''} \mathring{w}_{i''}$, $\mathring{n}_{i'} = \delta_{i'}^{i''} \mathring{n}_{i''}$. Introducing $g_{i'}, h_{a'}$ into (39) for given $\mathring{g}_{i''}, \mathring{h}_{a''}$, we can determine four values from eight ones, $\mathring{\mathbf{e}}_{i''}^{a''}$ and $\mathring{\mathbf{e}}_{a''}^{i''}$.

4.3.3 Nonholonomic Einstein spaces and (pseudo) Finsler variables

We summarize the main steps which allows us to transform a (pseudo) Finsler d-metric into a general (pseudo) Riemannian one and then to relate both such d-metrics to an exact solution of the Einstein equations. Of course, such geometric/physical models became equivalent if they are performed for the same canonical d-connection and/or Levi-Civita connection.

1. Let consider a solution for (non)holonomic Einstein spaces with a canonical d-metric:

$$\begin{aligned}\mathring{\mathbf{g}} &= \mathring{g}_i dx^i \otimes dx^i + \mathring{h}_a (dy^a + \mathring{N}_j^a dx^j) \otimes (dy^a + \mathring{N}_i^a dx^i) \\ &= \mathring{g}_i e^i \otimes e^i + \mathring{h}_a \mathring{\mathbf{e}}^a \otimes \mathring{\mathbf{e}}^a = \mathring{g}_{i'' j''} e^{i''} \otimes e^{j''} + \mathring{h}_{a'' b''} \mathring{\mathbf{e}}^{a''} \otimes \mathring{\mathbf{e}}^{b''}\end{aligned}$$

related to an arbitrary (pseudo) Riemannian metric with transforms of type (38).

2. We chose on \mathbf{V} a fundamental (pseudo) Finsler function $F = {}^3F(x^i, v) + {}^4F(x^i, y)$ inducing canonically a d-metric of type

$$\begin{aligned}\mathbf{f} &= f_i dx^i \otimes dx^i + f_a (dy^a + {}^c N_j^a dx^j) \otimes (dy^a + {}^c N_i^a dx^i), \\ &= f_i e^i \otimes e^i + f_a {}^c \mathbf{e}^a \otimes {}^c \mathbf{e}^a\end{aligned}$$

determined by data $\mathbf{f}_{\alpha\beta} = \left[f_i, f_a, {}^c N_j^a \right]$ in a canonical N-elongated base ${}^c \mathbf{e}^\alpha = (dx^i, {}^c \mathbf{e}^a = dy^a + {}^c N_i^a dx^i)$.

3. From formulas (40) with $N_{i'}^{a'} = \mathring{N}_{i'}^{a'}$ and $\mathbf{e}^{\alpha'} = \mathring{\mathbf{e}}^{\alpha'}$, we obtain

$$g_{i'} = f_{i'} \left(\frac{\mathring{w}_{i'}}{c w_{i'}} \right)^2 \frac{h_{3'}}{f_{3'}}, \quad g_{i'} = f_{i'} \left(\frac{\mathring{n}_{i'}}{c n_{i'}} \right)^2 \frac{h_{4'}}{f_{4'}}.$$

Both formulas are compatible if $\mathring{w}_{i'}$ and $\mathring{n}_{i'}$ are constrained (this is possible if we chose (17) and (19)) to satisfy the conditions

$$\Theta_{1'} = \Theta_{2'} = \Theta,$$

where $\Theta_{i'} = \left(\frac{\mathring{w}_{i'}}{c w_{i'}} \right)^2 \left(\frac{\mathring{n}_{i'}}{c n_{i'}} \right)^2$, and $\Theta = \left(\frac{\mathring{w}_{1'}}{c w_{1'}} \right)^2 \left(\frac{\mathring{n}_{1'}}{c n_{1'}} \right)^2 = \left(\frac{\mathring{w}_{2'}}{c w_{2'}} \right)^2 \left(\frac{\mathring{n}_{2'}}{c n_{2'}} \right)^2$. Having computed Θ , we can define

$$g_{i'} = \left(\frac{\mathring{w}_{i'}}{c w_{i'}} \right)^2 \frac{f_{i'}}{f_{3'}} \text{ and } h_{3'} = h_{4'} \Theta,$$

where (in this case) there is not summing on indices. So, we constructed the data $g_{i'}$, $h_{a'}$ and $w_{i'}$, $n_{j'}$.

4. We can construct $\mathring{\mathbf{e}}_{i''}^{a'}$ and $\mathring{\mathbf{e}}_{a''}^{i'}$ as any nontrivial solutions of

$$\mathring{g}_{i''} = g_{i''} + h_{a'} \left(\mathring{\mathbf{e}}_{i''}^{a'} \right)^2, \quad \mathring{h}_{a''} = g_{i'} \left(\mathring{\mathbf{e}}_{a''}^{i'} \right)^2 + h_{a''}, \quad \mathring{N}_{i''}^{a''} = N_{i''}^{a''}.$$

For instance, we can take

$$\begin{aligned} \mathring{\mathbf{e}}_{1''}^{3'} &= \pm \sqrt{|(\mathring{g}_{1''} - g_{1''})/h_{3'}|}, \quad \mathring{\mathbf{e}}_{2''}^{3'} = 0, \quad \mathring{\mathbf{e}}_{i''}^{4'} = 0 \\ \mathring{\mathbf{e}}_{a''}^{1'} &= 0, \quad \mathring{\mathbf{e}}_{3''}^{2'} = 0, \quad \mathring{\mathbf{e}}_{4''}^{2'} = \pm \sqrt{|(\mathring{h}_{4''} - h_{4''})/g_{2'}|} \end{aligned}$$

and finally compute

$$e_{1'}^{1'} = \pm \sqrt{\left| \frac{f_1}{g_{1'}} \right|}, \quad e_{2'}^{2'} = \pm \sqrt{\left| \frac{f_2}{g_{2'}} \right|}, \quad e_{3'}^{3'} = \pm \sqrt{\left| \frac{f_3}{h_{3'}} \right|}, \quad e_{4'}^{4'} = \pm \sqrt{\left| \frac{f_4}{h_{4'}} \right|}.$$

We note that we defined a sequence of two nonholonomic deformations from \mathbf{f} to $\mathring{\mathbf{g}}$ and inversely. The above geometric constructions are outlined in Table 1.

The goal of this section was to prove that for any model of (pseudo) Finsler gravity induced by a generating function of type $F = {}^3F(\xi, \vartheta, \varphi) + {}^4F(\xi, \vartheta, \varphi)$ there are exact solutions with rotoid symmetry, of type (16),

\mathbf{f} $\{\mathbf{f}_{\alpha\beta}\}$	\leftrightarrow	\mathbf{g} $\{\mathbf{f}_{\alpha\beta} = \mathbf{e}^{\alpha'}_{\alpha} \mathbf{e}^{\beta'}_{\beta} \mathbf{g}_{\alpha'\beta'}\}$	\leftrightarrow	$\hat{\mathbf{g}}$ $\{\mathbf{g}_{\alpha'\beta'} = \hat{\mathbf{e}}^{\alpha''}_{\alpha'} \hat{\mathbf{e}}^{\beta''}_{\beta'} \hat{\mathbf{g}}_{\alpha''\beta''}\}$
		$\mathbf{e}^{\alpha'}_{\alpha} = [e^{i'}_{i}, \hat{\mathbf{e}}^{a'}_{a}]$		$\begin{aligned} \hat{\mathbf{e}}^{a'}_{i''} &= \begin{bmatrix} \hat{\mathbf{e}}^{3'}_{1''}; \hat{\mathbf{e}}^{a'}_{a''} = \delta^{a'}_{a''} \\ \hat{\mathbf{e}}^{4'}_{i''} = \hat{\mathbf{e}}^{3'}_{2''} = 0 \end{bmatrix} \\ \hat{\mathbf{e}}^{i'}_{a''} &= \begin{bmatrix} \hat{\mathbf{e}}^{i'}_{i''} = \delta^{i'}_{i''}; \hat{\mathbf{e}}^{2'}_{4''} \\ \hat{\mathbf{e}}^{1'}_{a''} = \hat{\mathbf{e}}^{2'}_{3''} = 0 \end{bmatrix} \\ \hat{\mathbf{e}}^{\alpha}_{\alpha''} &= [\hat{\mathbf{e}}^{a'}_{i''}, \hat{\mathbf{e}}^{i'}_{a''}] \end{aligned}$
$f_{ij} = \text{diag}\{f_i\}$ $f_{ab} = \text{diag}\{f_a\}$ ${}^c N_i^a = \begin{cases} {}^c w_i \\ {}^c n_i \end{cases}$ ${}^c \mathbf{e}^{\alpha} = (dx^i, {}^c \mathbf{e}^a)$		$g_{i'j'} = \text{diag}\{g_{i'}\}$ $h_{a'b'} = \text{diag}\{h_{a'}\}$ $N_{i'}^{a'} = \hat{N}_{i'}^{a'}$ $\mathbf{e}^{\alpha'} = (dx^{i'}, \hat{\mathbf{e}}^{a'})$		$\hat{g}_{i''j''} = \text{diag}\{\hat{g}_{i''}\}$ $\hat{h}_{a''b''} = \text{diag}\{\hat{h}_{a''}\}$ $\hat{N}_{i''}^{a''} = \begin{cases} \hat{w}_{i''} \\ \hat{n}_{i''} \end{cases}$ $\hat{\mathbf{e}}^{\alpha''} = (\hat{\mathbf{e}}^{i''}, \hat{\mathbf{e}}^{a''})$

Table 1: Nonholonomic deformations of (pseudo) Finsler metrics into (pseudo) Riemannian/ Einstein ones.

for Einstein equations with nontrivial cosmological constant. In the limit $\varepsilon \rightarrow 0$ for $\hat{\mathbf{g}}$, the elaborated scheme of two nonholonomic transforms allows us to rewrite the Schwarzschild solution as a (pseudo) Finsler metric $\mathbf{f}(x, y)$. Haven chosen to define our gravity theory on a N-anholonomic manifold, we say that the the Schwarzschild spacetime is parametrized in (nonholonomic) Finsler variables.

A construction similar to that on nonholonomic (pseudo) Riemannian spaces holds true for Finsler gravity theories on tangent bundles. In such a case, the variables y^a must be interpreted as "velocities" and the fundamental geometric objects (the metric and N- and d-connections) will depend on such tangent vectors components. As a natural Schwarzschild like generalization of $\hat{\mathbf{g}}$ would be to chose a d-metric $\mathbf{g}_{\alpha'\beta'} = [g_{i'j'}, h_{a'b'}, N_{i'}^{a'}]$ (1) included in a scheme $\mathbf{f} \leftrightarrow \mathbf{g} \leftrightarrow \hat{\mathbf{g}}$ when the canonical d-connection $\hat{\Gamma}_{\alpha\beta}^{\gamma}$ (A.25) is for a solution of nonholonomic vacuum Einstein equations with h_4 and h_3 defined respectively by $b^2 = q$ and $(b^*)^2 = [(\sqrt{|q|})^*]^2$ introduced in (28) but with general N-connection coefficients (19). Such configurations seem to be stable and define (nonholonomic) black hole objects in (pseudo) Finsler gravity (we have to chose correspondingly the integration functions ${}^1n_i(\xi, \vartheta)$, ${}^2n_i(\xi, \vartheta)$ and the coefficients $w_1(\xi, \vartheta, \varphi)$ and adapt the proof for "black ellipsoids" from [18, 19, 9]).

5 Discussion and Conclusions

This paper was primarily motivated by the question of how black hole solutions can be constructed in Finsler gravity theories and if such geometric objects may have relation to the physics of black holes in general relativity and generalizations. To the best of our knowledge, such questions have not yet been addressed in the literature.

There are also other motivations to study possible black hole structures and their nonholonomic deformations on (pseudo) Riemannian spacetimes and gravity models on tangent bundles:

1. To analyze physical consequences of nonholonomic frame constraints on the dynamics of gravitational fields and local anisotropies of fundamental field interactions induced by spacetime nonholonomic gravitational distributions and/or nonlinear self-polarizations of gravitational fields.
2. To provide additional arguments on viability of Finsler like gravity theories; if such models admit black hole objects which are non-trivially related to those in general relativity, this may help a better understanding of stationary gravitational configurations and their modelling by Lagrange-Finsler geometries.

Our constructions have completed a qualitative understanding of a class of exact solutions in the Einstein and Finsler gravity theories which for certain small values of parametric nonholonomic deformations contain stable ellipsoid configurations and new classes of black hole objects. There were encompassed all possible values of cosmological constant for solutions with generic off-diagonal metrics and two classes of linear connections (the canonical distinguished connection and the Levi-Civita one). The solutions were generated following the anholonomic frame method (see reviews of such geometric methods and results in Refs. [7, 8, 10]). Certain features of these solutions are shared, while others differ or can be modelled in certain limits of a small parameter and for some types of generating/integrating functions. For instance, we positively get black hole solutions with ellipsoidal symmetry for certain small values of eccentricity, but dependence on cosmological constant plays not a smooth character because of nonlinear interactions and nonholonomic constraints. At the most basic level, we have to introduce locally anisotropic polarizations of masses in order to get self-consistent and stable gravitational configurations.

In the context of interactions of Finsler black hole solutions with non-linear waves (we have chosen the example of solitonic waves), our geometric method allows us to include them both as generic off-diagonal terms and/or in the so-called "vertical" part of the metrics as small and not small deformations of original black holes spacetimes. We can consider various types of asymptotic conditions and nonholonomic constraints and define stationary stable configurations.

In another context of nonholonomic Einstein spacetimes modelled on (pseudo) Riemannian/ Finsler manifolds/bundles, we found that the same classes of black hole solutions, rotoids and/or solitons can be derived in all metric compatible gravity theories. The results of this paper have lead to an overview of the main qualitative features common to static solutions in general relativity and their stationary modifications for Finsler like theories. These confirm and expand our results and knowledge on existing/possible generalizations for black hole solutions in string/brane models, noncommutative gravity, nonholonomic Ricci flow theory etc, see examples and discussions in Refs. [13, 17, 59, 60, 61, 62, 63, 64, 65].

While the present work is concerned with four dimensional nonholonomic configurations in general relativity and the (pseudo) Finsler model with canonical distinguished connection, the anoholonomic frame method can be applied various types of connections in higher and lower dimension spacetimes [14, 15, 16], in quantum gravity [22, 23, 24, 26, 27, 28, 29] and supersymmetric/ superstring generalizations [39, 40].

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A Pseudo-Finsler Geometries on Tangent Bundles/ Einstein Manifolds

We provide a summary on (pseudo) Finsler geometries modelled on tangent bundles and (or) nonholonomic (pseudo) Riemannian manifolds, see details in Refs. [50, 8, 10]. For simplicity, we restrict our considerations for four dimensional (4-d) spacetime models with nonholonomic distributions (splitting) of type 2+2. There are emphasized the key constructions with Finsler like variables on (pseudo) Riemannian spacetimes and, inversely, re-definition of fundamental geometric objects on Finsler spaces as non-holonomic ones in Riemann geometry (we shall omit the term "pseudo" if that will not result in ambiguities).

A.1 The canonical N-connection and d-metric structures

A standard two dimensional Finsler space ${}^2F(M, F(x, y))$ is defined on a tangent bundle TM , where $M, \dim M = 2$, is the base manifold being differentiable of class C^∞ . One considers: 1) a fundamental real Finsler (generating) function $F(u) = F(x, y) = F(x^i, y^a) > 0$ if $y \neq 0$ and homogeneous of type $F(x, \lambda y) = |\lambda|F(x, y)$, for any nonzero $\lambda \in \mathbb{R}$, with positively definite Hessian

$$f_{ab} = \frac{1}{2} \frac{\partial^2 F^2}{\partial y^a \partial y^b}, \quad (\text{A.1})$$

when $\det |f_{ab}| \neq 0$. In order to state completely a geometric model (Finsler geometry) on TM , we have to chose 2) a nonlinear connection (N-connection) structure \mathbf{N} on TM defined by a nonholonomic distribution (Whitney sum)

$$TTM = hTM \oplus vTM \quad (\text{A.2})$$

into conventional horizontal (h), hTM , and vertical (v), vTM , subspaces and 3) a linear connection structure which is convenient to be defined in N-adapted form, i.e. preserving the splitting (A.2), called distinguished connection (in brief, d-connection) and denoted $\mathbf{D} = (hD, vD)$, or $\mathbf{D}_\alpha = (D_i, D_a)$.

Instead of a tangent bundle TM , we can model a Finsler geometry on a 4-d manifold 4V (of necessary smooth class) if we consider that such a manifold is enabled with a nonholonomic distribution of type (A.2). For instance, we can consider 4V as a Riemannian manifold of signature $(+, +, +, +)$, on which we chose any convenient system of reference (tetradic structure, equivalently, vierbein structure), local coordinates and any function $F(u) = F(x, y)$, with a conventional splitting into h- and v-coordinates, subjected to satisfy the conditions requested for a fundamental Finsler function. Fixing an explicit function $F(u)$, we state a Finsler geometry model (equivalently, configuration / structure on 4V).

In order to model a pseudo-Finsler geometry on TM , or 4V , we have to "relax" the condition that Hessian (A.1) is positively definite and consider that it can be negative, or even degenerate. In a fixed point, we may have local pseudo-Euclidean metrics, on h- and/or v-subspaces. We can use the above definition of Finsler space for tangent bundles/ manifolds enabled with local coordinates and frames parametrized in such a form that one of them is proportional to the imaginary unity as we explained in footnote 3, for instance $y^4 = i\tilde{y}^4$ for a real coordinate \tilde{y}^4 , or $x^2 = i\tilde{x}^2$ for a real coordinate \tilde{x}^2 (we can consider pseudo-Euclidean signatures with imaginary unity for any local coordinate). In general, we say that we model a

(pseudo) Finsler geometry 2F depending on signature (locally Euclidean, or pseudo-Euclidean), with a chosen coordinate parametrizations on a general nonholonomic manifold \mathbf{V} (we can consider $\mathbf{V} = TM$, or $\mathbf{V} = {}^4V$) with, or without, imaginary unity. So, pseudo-Finsler spaces can be naturally modelled on pseudo-Riemannian manifolds.

In local form, a N-connection on \mathbf{V} is given by its coefficients $N_i^a(u)$, when

$$\mathbf{N} = N_i^a(u) dx^i \otimes \frac{\partial}{\partial y^a}. \quad (\text{A.3})$$

It states a frame (vielbein) structure which is linear on N-connection coefficients,

$$\mathbf{e}_\nu = \left(\mathbf{e}_i = \frac{\partial}{\partial x^i} - N_i^a(u) \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a} \right), \quad (\text{A.4})$$

$$\mathbf{e}^\mu = (e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a(u) dx^i). \quad (\text{A.5})$$

The vielbeins (A.5) satisfy the nonholonomy relations

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = w_{\alpha\beta}^\gamma \mathbf{e}_\gamma \quad (\text{A.6})$$

with (antisymmetric) nontrivial anholonomy coefficients $w_{ia}^b = \partial_a N_i^b$ and $w_{ji}^a = \Omega_{ij}^a$, where

$$\Omega_{ij}^a = \mathbf{e}_j (N_i^a) - \mathbf{e}_i (N_j^a) \quad (\text{A.7})$$

define the coefficients of N-connection curvature. The particular holonomic/integrable case is selected by the integrability conditions $w_{\alpha\beta}^\gamma = 0$.¹⁰

A N-anholonomic manifold is a (nonholonomic) manifold enabled with N-connection structure (A.2). The properties of a N-anholonomic manifold are determined by N-adapted bases (A.4) and (A.5). A geometric object is N-adapted (equivalently, distinguished), i.e. a d-object, if it can be defined by components adapted to the splitting (A.2) (one uses terms d-vector, d-form, d-tensor). For instance, a d-vector is represented as $\mathbf{X} = X^\alpha \mathbf{e}_\alpha = X^i \mathbf{e}_i + X^a e_a$ and a one d-form $\tilde{\mathbf{X}}$ (dual to \mathbf{X}) is represented as $\tilde{\mathbf{X}} = X_\alpha \mathbf{e}^\alpha = X_i e^i + X_a e^a$.¹¹

For a (pseudo) Finsler space, it is possible to construct such a canonical (Cartan) N-connection ${}^c\mathbf{N} = \{ {}^cN_i^a \}$ completely defined by an effective

¹⁰We use boldface symbols for spaces (and geometric objects on such spaces) enabled with N-connection structure.

¹¹We can redefine equivalently the geometric constructions for arbitrary frame and coordinate systems; the N-adapted constructions allow us to preserve the h- and v-splitting.

Lagrangian $L = F^2$ in such a form that the corresponding semi-spray configuration is defined by nonlinear geodesic equations being equivalent to the Euler–Lagrange equations for L (see details, for instance, in Refs. [50, 8, 10]; for "pseudo" configurations, this mechanical analogy is a formal one, with some "imaginary" coordinates). One defines

$${}^cN_i^a = \frac{\partial G^a}{\partial y^{2+i}}, \quad (\text{A.8})$$

for

$$G^a = \frac{1}{4} f^{a\ 2+i} \left(\frac{\partial^2 L}{\partial y^{2+i} \partial x^k} y^{2+k} - \frac{\partial L}{\partial x^i} \right), \quad (\text{A.9})$$

where f^{ab} is inverse to f_{ab} (A.1) and respective contractions of h - and v -indices, i, j, \dots and a, b, \dots , are performed following the rule: we can write, for instance, an up v -index a as $a = 2 + i$ and contract it with a low index $i = 1, 2$. Briefly, for spaces of even dimension, we can write y^i instead of y^{2+i} , or y^a .

The values (A.1), (A.8) and (A.9) allow us to define the canonical Sasaki type metric (d-metric, equivalently, metric d-tensor)

$$\begin{aligned} \mathbf{f} &= f_{ij} dx^i \otimes dx^j + f_{ab} {}^c\mathbf{e}^a \otimes {}^c\mathbf{e}^b, \\ {}^c\mathbf{e}^a &= dy^a + {}^cN_i^a dx^i, \end{aligned} \quad (\text{A.10})$$

where $f_{ij} = f_{2+i\ 2+j}$ for (pseudo) Finsler spaces with (pseudo) Euclidean local parametrizations using "imaginary" unity. With respect to a local dual coordinate basis $du^\alpha = (dx^i, dy^a)$, this "total" (pseudo) Finsler metric is parametrized in the form

$$f = \underline{f}_{\alpha\beta}(u) du^\alpha \otimes du^\beta, \quad (\text{A.11})$$

where

$$\underline{f}_{\alpha\beta} = \begin{bmatrix} f_{ij} + {}^cN_i^a {}^cN_j^b f_{ab} & {}^cN_j^e f_{ae} \\ {}^cN_i^e f_{be} & f_{ab} \end{bmatrix}. \quad (\text{A.12})$$

This is also a (pseudo) Riemannian metric (A.12) with coefficients induced canonically by a Finsler fundamental function $F(u)$ following formulas (A.1) and (A.8). Such a metric is generic off-diagonal because, in general, it can not be diagonalized by coordinate transforms.

We conclude our geometric constructions with two important remarks:

The metric and nonholonomic structure of (pseudo) Finsler geometry, defined by a fundamental Finsler (generating) function $F(x, y)$ can be modelled canonically on a tangent bundle/ N-anholonomic manifold by a Sasaki

type metric (A.10), which is equivalent to a local (pseudo) Riemannian (generic off-diagonal) metric (A.11) with coefficients parametrized in the form (A.12). General frame transforms of type $\mathbf{f}_{\alpha\beta} = \mathbf{g}_{\alpha'\beta'} e^{\alpha'}_{\alpha} e^{\beta'}_{\beta}$ (2) "hide" the Finsler structure and "mix" \mathbf{f} into a general (pseudo) Riemannian metric \mathbf{g} (1) even if we work on a tangent bundle.

Inversely, for any (pseudo) Riemannian metric \mathbf{g} (1), on a tangent bundle/ manifold, we can chose formally a necessary type $F(x, y)$ inducing a formal nonholonomic 2+2 splitting with N-connection structure ${}^c N_i^a$ (A.8) and associated nonholonomic frames of type (A.4) and (A.5) with such canonical N-coefficients. Taking any real solution $e^{\alpha'}_{\alpha}$ of algebraic equations (2), for prescribed values of $\mathbf{f}_{\alpha\beta}$ and $\mathbf{g}_{\alpha'\beta'}$, we represent locally \mathbf{g} as a (pseudo) Finsler metric \mathbf{f} . One say that we have chosen Finsler / nonholonomic variables, or coordinates, on a (pseudo) Riemann manifold. It is a matter of convenience with what type of variables/ coordinates we work on such spacetimes.

A.2 Distinguished connections on (pseudo) Finsler spaces

Having stated canonical metric and nonholonomic (N-connection) structures of a (pseudo) Finsler space, it is necessary to chose what type of linear connection we are going to use. Of course, we can always construct for any \mathbf{f} its Levi-Civita connection ${}^F \nabla$ as in usual (pseudo) Riemann geometry. But such a linear connection is not N-adapted because it does not preserve the h- and v-splitting (A.2) by general coordinate transforms. That why in (pseudo) Finsler geometry it is preferred to work with a different class of linear connections which allows us to perform geometric constructions in a form adapted to the N-connection splitting.

A.2.1 Torsion and curvature of d-connections

A distinguished connection (d-connection) \mathbf{D} on a (pseudo) Finsler space \mathbf{V} is a linear connection conserving under parallelism the Whitney sum (A.2). For any d-vector \mathbf{X} , there is a decomposition of \mathbf{D} into h- and v-covariant derivatives,

$$\mathbf{D}\mathbf{X} \doteq \mathbf{X} \rfloor \mathbf{D} = hX \rfloor \mathbf{D} + vX \rfloor \mathbf{D} = Dh_X + D_v X = hD_X + vD_X. \quad (\text{A.13})$$

The symbol " \rfloor " in (A.13) is the interior product induced by a metric \mathbf{g} (1) (for (pseudo) Finsler spaces we have to use a metric ${}^F \mathbf{g}$ (A.11), equivalently, by a d-metric (A.10)). The N-adapted components $\Gamma^\alpha_{\beta\gamma}$ of a d-connection

$\mathbf{D}_\alpha = (\mathbf{e}_\alpha \rfloor \mathbf{D})$ are defined by equations

$$\mathbf{D}_\alpha \mathbf{e}_\beta = \Gamma_{\alpha\beta}^\gamma \mathbf{e}_\gamma, \text{ or } \Gamma_{\alpha\beta}^\gamma(u) = (\mathbf{D}_\alpha \mathbf{e}_\beta) \rfloor \mathbf{e}^\gamma. \quad (\text{A.14})$$

The N-adapted splitting into h- and v-covariant derivatives is stated by $h\mathbf{D} = \{\mathbf{D}_k = (L_{jk}^i, L_{bk}^a)\}$ and $v\mathbf{D} = \{\mathbf{D}_c = (C_{jc}^i, C_{bc}^a)\}$, where $L_{jk}^i = (\mathbf{D}_k \mathbf{e}_j) \rfloor \mathbf{e}^i$, $L_{bk}^a = (\mathbf{D}_k e_b) \rfloor \mathbf{e}^a$, $C_{jc}^i = (\mathbf{D}_c \mathbf{e}_j) \rfloor \mathbf{e}^i$, $C_{bc}^a = (\mathbf{D}_c e_b) \rfloor \mathbf{e}^a$. The components

$$\Gamma_{\alpha\beta}^\gamma = (L_{jk}^i, L_{bk}^a, C_{jc}^i, C_{bc}^a) \quad (\text{A.15})$$

completely define a d-connection \mathbf{D} . We shall write conventionally that $\mathbf{D} = (hD, vD)$, or $\mathbf{D}_\alpha = (D_i, D_a)$, with $hD = (L_{jk}^i, L_{bk}^a)$ and $vD = (C_{jc}^i, C_{bc}^a)$, see (A.14).

The simplest way to perform computations with d-connections is to use the N-adapted differential 1-form

$$\Gamma_\beta^\alpha = \Gamma_{\beta\gamma}^\alpha \mathbf{e}^\gamma \quad (\text{A.16})$$

with the coefficients defined with respect to (A.5) and (A.4). For instance, torsion of \mathbf{D} ,

$$\mathcal{T}^\alpha \doteq \mathbf{D}\mathbf{e}^\alpha = d\mathbf{e}^\alpha + \Gamma_\beta^\alpha \wedge \mathbf{e}^\beta, \quad (\text{A.17})$$

can be computed to have (N-adapted) nontrivial antisymmetric d-torsion $\mathbf{T}^\alpha_{\beta\gamma} = \{T_{jk}^i, T_{ja}^i, T_{ji}^a, T_{bi}^a, T_{bc}^a\}$ coefficients

$$\begin{aligned} T_{jk}^i &= L_{jk}^i - L_{kj}^i, \quad T_{ja}^i = -T_{aj}^i = C_{ja}^i, \quad T_{ji}^a = \Omega_{ji}^a, \\ T_{bi}^a &= -T_{ib}^a = \frac{\partial N_i^a}{\partial y^b} - L_{bi}^a, \quad T_{bc}^a = C_{bc}^a - C_{cb}^a. \end{aligned} \quad (\text{A.18})$$

Similarly, one computes the nontrivial N-adapted components of curvature of d-connection \mathbf{D} , d-curvature

$$\mathbf{R}^\alpha_{\beta\gamma\delta} = \{R_{hjk}^i, R_{bjk}^a, R_{jka}^i, R_{bka}^c, R_{jbc}^i, R_{bcd}^a\},$$

$$\mathcal{R}^\alpha_\beta \doteq \mathbf{D}\Gamma_\beta^\alpha = d\Gamma_\beta^\alpha - \Gamma_\beta^\gamma \wedge \Gamma_\gamma^\alpha = \mathbf{R}^\alpha_{\beta\gamma\delta} \mathbf{e}^\gamma \wedge \mathbf{e}^\delta, \quad (\text{A.19})$$

when

$$\begin{aligned} R_{hjk}^i &= e_k L_{hj}^i - e_j L_{hk}^i + L_{hj}^m L_{mk}^i - L_{hk}^m L_{mj}^i - C_{ha}^i \Omega_{kj}^a, \\ R_{bjk}^a &= e_k L_{bj}^a - e_j L_{bk}^a + L_{bj}^c L_{ck}^a - L_{bk}^c L_{cj}^a - C_{bc}^a \Omega_{kj}^c, \\ R_{jka}^i &= e_a L_{jk}^i - D_k C_{ja}^i + C_{jb}^i T_{ka}^b, \\ R_{bka}^c &= e_a L_{bk}^c - D_k C_{ba}^c + C_{bd}^c T_{ka}^d, \\ R_{jbc}^i &= e_c C_{jb}^i - e_b C_{jc}^i + C_{jb}^h C_{hc}^i - C_{jc}^h C_{hb}^i, \\ R_{bcd}^a &= e_d C_{bc}^a - e_c C_{bd}^a + C_{bc}^e C_{ed}^a - C_{bd}^e C_{ec}^a. \end{aligned} \quad (\text{A.20})$$

Contracting respectively the components of (A.20), one proves that the Ricci tensor $\mathbf{R}_{\alpha\beta} \doteq \mathbf{R}^{\tau}_{\alpha\beta\tau}$ is characterized by h- v-components, i.e. the Ricci d-tensor $\mathbf{R}_{\alpha\beta} = \{R_{ij}, R_{ia}, R_{ai}, R_{ab}\}$,

$$R_{ij} \doteq R^k_{ijk}, \quad R_{ia} \doteq -R^k_{ika}, \quad R_{ai} \doteq R^b_{aib}, \quad R_{ab} \doteq R^c_{abc}; \quad (\text{A.21})$$

for arbitrary d-connections \mathbf{D} , this tensor is not symmetric, i.e. $\mathbf{R}_{\alpha\beta} \neq \mathbf{R}_{\beta\alpha}$. In order to define the scalar curvature of a d-connection \mathbf{D} it is necessary to involve a d-metric structure \mathbf{g} on \mathbf{V} ,

$${}^s\mathbf{R} \doteq \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab}, \quad (\text{A.22})$$

summarizing the h- and v-components of (A.21) contracted with the coefficients of a metric being inverse to \mathbf{g} (1) (for (pseudo) Finsler spaces we have to use a metric ${}^F\mathbf{g}$ (A.11), equivalently, by a d-metric (A.10)).

The Einstein d-tensor is by definition

$$\mathbf{E}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} {}^s\mathbf{R}. \quad (\text{A.23})$$

This d-tensor defines a nonholonomic Einstein configuration for a d-connection \mathbf{D} which is alternative to the standard Einstein tensor constructed from the Levi-Civita connection.¹²

A.2.2 Canonical (pseudo) Finsler d-connections and the Levi-Civita connection

For geometric and physical applications, it is more convenient to consider d-connections \mathbf{D} which are metric compatible (metrical d-connections),

$$\mathbf{D}\mathbf{g} = \mathbf{0}, \quad (\text{A.24})$$

with $D_j g_{kl} = 0, D_a g_{kl} = 0, D_j h_{ab} = 0, D_a h_{bc} = 0$.

For any d-metric \mathbf{g} on a N-anholonomic manifold \mathbf{V} , there is a unique metric canonical d-connection $\hat{\mathbf{D}}$ satisfying the conditions $\hat{\mathbf{D}}\mathbf{g} = \mathbf{0}$ and with

¹²The Levi-Civita connection and related Christoffel symbols are the standard geometric objects used in general relativity; nevertheless, one must be emphasized that the Einstein theory can be formulated equivalently in terms of any linear connection if such a connection is completely determined by the (pseudo) Riemannian metric structure and, for instance, any prescribed values determining nonholonomic distributions. Using distortions of linear connections (see subsection A.2.2), we can re-express all fundamental physical values and equations in general relativity in terms of certain d-objects and, inversely, in terms of standard (pseudo) Riemannian, Levi-Civita, tensor calculus.

vanishing of "pure" horizontal and vertical torsion coefficients, i. e. $\hat{T}_{jk}^i = 0$ and $\hat{T}_{bc}^a = 0$, see formulas (A.18). Locally, we can write that $\hat{\mathbf{T}}_{\alpha\beta}^\gamma = (\hat{L}_{jk}^i, \hat{L}_{bk}^a, \hat{C}_{jc}^i, \hat{C}_{bc}^a)$ with

$$\begin{aligned}\hat{L}_{jk}^i &= \frac{1}{2}g^{ir}(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \\ \hat{L}_{bk}^a &= e_b(N_k^a) + \frac{1}{2}h^{ac}\left(e_k h_{bc} - h_{dc} e_b N_k^d - h_{db} e_c N_k^d\right), \\ \hat{C}_{jc}^i &= \frac{1}{2}g^{ik}e_c g_{jk}, \quad \hat{C}_{bc}^a = \frac{1}{2}h^{ad}(e_c h_{bd} + e_c h_{cd} - e_d h_{bc}).\end{aligned}\tag{A.25}$$

We emphasize that in general $\hat{T}_{ja}^i, \hat{T}_{ji}^a$ and \hat{T}_{bi}^a are not zero, but such non-trivial components of torsion are induced by some coefficients of a general off-diagonal metric $\mathbf{g}_{\alpha\beta}$, see explicit formulas in Refs. [8, 10].

Similar formulas holds true, for instance, for the Levi-Civita linear connection $\nabla = \{ {}_1\Gamma_{\beta\gamma}^\alpha \}$ which is uniquely defined by a metric structure by conditions ${}_1\mathcal{T} = 0$ and $\nabla \mathbf{g} = 0$. This connection is largely used in (pseudo) Riemannian geometry. Any geometric construction for the canonical d-connection $\hat{\mathbf{D}}$ can be re-defined equivalently into a similar one with the Levi-Civita connection following formula

$${}_1\Gamma_{\alpha\beta}^\gamma = \hat{\mathbf{T}}_{\alpha\beta}^\gamma + {}_1Z_{\alpha\beta}^\gamma,\tag{A.26}$$

where the distortion tensor ${}_1Z_{\alpha\beta}^\gamma$ is constructed in a unique form from the coefficients of a metric $\mathbf{g}_{\alpha\beta}$,

$$\begin{aligned}{}_1Z_{jk}^a &= -C_{jb}^i g_{ik} h^{ab} - \frac{1}{2}\Omega_{jk}^a, \quad {}_1Z_{bk}^i = \frac{1}{2}\Omega_{jk}^c h_{cb} g^{ji} - \Xi_{jk}^{ih} C_{hb}^j, \\ {}_1Z_{bk}^a &= {}^\pm \Xi_{cd}^{ab} {}^\circ L_{bk}^c, \quad {}_1Z_{kb}^i = \frac{1}{2}\Omega_{jk}^a h_{cb} g^{ji} + \Xi_{jk}^{ih} C_{hb}^j, \quad {}_1Z_{jk}^i = 0, \\ {}_1Z_{jb}^a &= -{}^\pm \Xi_{cb}^{ad} {}^\circ L_{dj}^c, \quad {}_1Z_{bc}^a = 0, \quad {}_1Z_{ab}^i = -\frac{g^{ij}}{2} [{}^\circ L_{aj}^c h_{cb} + {}^\circ L_{bj}^c h_{ca}],\end{aligned}\tag{A.27}$$

for $\Xi_{jk}^{ih} = \frac{1}{2}(\delta_j^i \delta_k^h - g_{jk} g^{ih})$, ${}^\pm \Xi_{cd}^{ab} = \frac{1}{2}(\delta_c^a \delta_d^b + h_{cd} h^{ab})$ and ${}^\circ L_{aj}^c = L_{aj}^c - e_a(N_j^c)$.

For both types of linear connections $\hat{\mathbf{D}}$ and ∇ , we can compute the curvature, torsion, Ricci and Einstein tensors and scalar curvature following respectively formulas (A.20), (A.18), (A.21) and (A.23) and (A.22). It is convenient, see details in [8], to label such objects in the form $\hat{\mathbf{R}}_{\beta\gamma\delta}^\alpha, \hat{\mathbf{T}}_{\beta\gamma}^\alpha, \hat{\mathbf{R}}_{\alpha\beta}$ and $\hat{\mathbf{E}}_{\alpha\beta}$ and ${}^s\hat{\mathbf{R}}$ and (correspondingly) ${}_1R_{\beta\gamma\delta}^\alpha, {}_1T_{\beta\gamma}^\alpha = 0, {}_1R_{\alpha\beta} = {}_1R_{\beta\alpha}$ and ${}_1E_{\alpha\beta}$ and ${}_1R$. Usually we consider that $\mathbf{R}_{\beta\gamma\delta}^\alpha, \mathbf{T}_{\beta\gamma}^\alpha, \mathbf{R}_{\alpha\beta}$ and $\mathbf{E}_{\alpha\beta}$ and ${}^s\mathbf{R}$ are for respective geometric values computed for a general d-connection

D. For simplicity, we omit "hats" on formulas, even they are for certain canonical values, if this does not result in ambiguities.

We note here that different linear connections, ∇ and $\hat{\mathbf{D}}$, are subjected to different transformation rules. It is possible to chose a nonholonomic distribution/frame when ${}_1Z^\gamma_{\alpha\beta} = 0$ and there is a formal equality of coefficients ${}_1\Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta}$ (for a particular frame of reference, we can construct exact solutions for a d-metric \mathbf{g} when $\hat{\mathbf{R}}_{\alpha\beta} = {}_1R_{\alpha\beta} = \lambda\mathbf{g}_{\beta\beta}$). In general, $\hat{\mathbf{R}}^\alpha_{\beta\gamma\delta} \neq {}_1R^\alpha_{\beta\gamma\delta}$ and ${}^s\hat{\mathbf{R}} \neq {}^sR$, if we perform (with possible mixing) nonholonomic deformations of linear connections of type (A.26) and frame transforms of type $\mathbf{e}_\alpha = e^{\alpha'}_\alpha \mathbf{e}_{\alpha'}$. Fixing a metric \mathbf{g} and N-connection/frame structure on \mathbf{V} , we can compute the nonholonomic deformations $\hat{\mathbf{R}}^\alpha_{\beta\gamma\delta} = {}_1R^\alpha_{\beta\gamma\delta} + {}_1\hat{\mathbf{Z}}^\alpha_{\beta\gamma\delta}$ induced by deformations (A.26). The d-tensor $\hat{\mathbf{R}}^\alpha_{\beta\gamma\delta}$ is similar to that for a Riemann–Cartan space but with that difference that the d-torsion $\hat{\mathbf{T}}^\alpha_{\beta\gamma}$ is completely defined by the metric tensor \mathbf{g} , as well the distortion tensors ${}_1Z^\gamma_{\alpha\beta}$ and ${}_1\hat{\mathbf{Z}}^\alpha_{\beta\gamma\delta}$. Contracting indices, we can compute distortions of the Ricci and Einstein tensors which are also defined in unique forms by d-metric coefficients for a fixed N-anholonomic structure. So, on a nonholonomic spacetime \mathbf{V} , we can work equivalently with both types of linear connections ∇ and $\hat{\mathbf{D}}$.

Different schools on Finsler geometry worked with different classes of linear connections, which (in general) are not metric compatible. The most cited monographs are [55, 45] and many times it is considered that (pseudo) Finsler geometry provides a more general, and alternative, spacetime geometry than that for (pseudo) Riemannian spaces. Here we also add various types of models with broken local Lorentz symmetry and anisotropies in phase spaces with momenta/velocities [47, 48, 53, 32, 20, 21], all on (co) tangent bundles have been elaborated using metric noncompatible connections.

In Refs. [49, 50], there are considered the so-called Kawaguchi metrization and Miron's procedure when for any given metric and Finsler connection can be computed all possible nonmetric and metric compatible d-connections. Such constructions are applied in modern nonsymmetric gravity and quantum gravity [25, 28, 29]. So, in Finsler geometry and modern gravity theories, one works not only with arbitrary N-adapted frame and coordinate transforms but also with deformation/ distortion of d-connection structures. For a prescribed type of nonholonomic distributions (i.e. for a chosen nonholonomic configuration), we can always say that some nonholonomic deformations/ distortions are generated by a fundamental Finsler

structure and related metric tensor. In our approach to Finsler geometry, generalizations and applications, we proved that the N-connection formalism and Finsler geometry methods can be applied also in classical and quantum (models) of Einstein gravity, see Refs. [8, 7, 9, 10, 27], which is related to a multi-connection formalism. In order to generate black hole like solutions for (pseudo) Finsler/ nonholonomic Einstein spaces, it is convenient to use the canonical d-connection $\hat{\Gamma}^\gamma_{\alpha\beta}$ (A.25) and its distortion to the Levi-Civita connection ${}_l\Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta} + {}_lZ^\gamma_{\alpha\beta}$ (A.26).

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